Berkeley Math Circle: Monthly Contest 7 Solutions

1. Recall that a number is *prime* if its divisors over the positive integers are only 1 and itself. For instance, one can verify that 3 is prime, but note that 15 is not prime because it is divisible by 3 and 5 in addition to 1 and itself.

   How many positive integers n are there such that n^2+1 and n^2+6 are both prime?

   SOLUTION. If $n > 1$ is odd, then n^2+1 is even and composite. If $n > 1$ is even, then n^2+6 is even and composite. Hence $n = 1$ is our only solution, as indeed $1^1+1 = 2$ and $1^2+6 = 7$ are both prime. Thus the answer is $\boxed{1}$.

2. An n-digit number is *narcissistic* if it equals the sum of the nth powers of its digits. For instance, the number 32164049650 is a narcissistic 11-digit number because

   $$32164049650 = 3^{11}+2^{11}+1^{11}+6^{11}+4^{11}+0^{11}+4^{11}+9^{11}+6^{11}+5^{11}+0^{11}.$$

   It turns out there are exactly two narcissistic 11-digit numbers under 40000000000. Let them be x and y. Determine the value of $|x-y|$.

   SOLUTION. Of course, up to reordering x and y are 32164049650 and 32164049651. Hence $|x-y| = \boxed{1}$.

3. In chess, a *queen* can move horizontally, vertically, or diagonally. A square of a chessboard is *attacked* by a queen if the queen can move there in one move.

   Let n be a positive integer such that $n \geq 6$. Is it possible to place $n-2$ queens on a $n \times n$ chessboard such that every empty square is attacked by at least one queen? You are given that for all $m \geq 4$, it is possible to place m queens on an $m \times m$ grid such that no two queens attack each other.

   SOLUTION. The answer is $\boxed{\text{yes}}$. Fix $n \geq 6$. As given, let $\mathcal{P}$ be a placement of $n-2$ queens on an $(n-2)\times(n-2)$ chessboard such that no two queens attack each other. Thus $\mathcal{P}$ has exactly one queen in every row and exactly one queen in every column. Let q be one of the queens in $\mathcal{P}$, and insert rows r_1 and r_2 and columns c_1 and c_2 on either side of q. This yields a $n \times n$ chessboard with a placement $\mathcal{P}'$ of $n-2$ queens. If a square s of $\mathcal{P}'$ is contained in a row r or column c other than the r_i, then r or c contains a queen from $\mathcal{P}$ attacking s. The only cases left to consider are the four squares given by intersecting the r_i with the c_i, but they must all be diagonally be attacked by q. We are done.

4. It is helpful to read the previous problem to obtain relevant background information.

   Let k be a positive integer satisfying $k \geq 4$, and let $m < \frac{k}{3}$. Show that one cannot place m queens on a $k \times k$ chessboard so that every empty square is attacked by at least one queen.

SOLUTION. Fix $k \geq 4$ and let $m < \frac{k}{3}$. Consider an arbitrary placement of m queens on a $k \times k$ chessboard. Let r and c be the number of distinct rows and columns of the chessboard, respectively, on which at least one queen is placed.

One can check that the number of empty squares attacked vertically by one or more of queens is $ck-m$, and the number of empty squares attacked horizontally that have not been accounted for already is $r(k-c)$. An arbitrarily placed queen diagonally attacks at most two squares in each column. Since c of these columns have already been accounted for, it follows that, out of the set of squares not included in our previous count, the queens diagonally attack at most $2m(k-c)$ empty squares.

It follows that the number of squares attacked by the queens is at most

$$(ck-m) + r(k-c) + 2m(k-c) = (2k-2c-1)m + (c+r)k - rc.$$

It suffices to show that this number is at most $k^2 - m$. Equivalently, it suffices to demonstrate the bound

$$2(k-c)m + (c+r)k < rc + k^2.$$

Since $m < \frac{k}{3}$, observe that

$$\begin{aligned}
\delta &= 2(k-c)m + (c+r)k - rc - k^2 \\
&< \frac{2(k-c)k}{3} + (c+r)k - rc - k^2 \\
&= -\frac{(k-3r)(k-c)}{3}.
\end{aligned}$$

Since $c \leq k \leq 3r$, it follows that $\delta < 0$, as desired.

5. One day at Evan Corporation LLC, developer Aerith writes a polynomial $P(x)$ on the whiteboard in the break room. Like everyone else, the polynomial is written so that the degrees of each term strictly decrease when reading from left to right. Unfortunately, CEO V. Enhance accidentally smudges part of the polynomial! Here is a snapshot of the current disarray of the whiteboard:

$$P(x) = 2x^4 \text{[redacted]} - 397x + 438.$$

Mr. Enhance asks three of Aerith's coworkers on the details of P.

- Ron recalls that $P(5) = 3$.
- Adi says that the squares of the roots of P add up to exactly 63.
- Leo says that P is divisible by $x^3 - 2x^2 - 17x - 42$.

Of these three statements, it turns out that exactly two of them are truthful. Determine the value of $P(-4)$.

SOLUTION. We deduce that

$$P(x) = 2x^4 + bx^3 + cx^2 - 397x + 438$$

for coefficients b and c.

Ron's claim implies that $2 \cdot 625 + 125b + 25c - 397 \cdot 5 + 438 = 3$, or equivalently

$$5b + c = 12.$$

Adi's claim can be interpreted using Vieta's formulas to say that $(-\frac{b}{2})^2 - 2(\frac{c}{2}) = 63$, which simplifies to having

$$b^2 - 4c = 252.$$

Leo's claim that P is divisible by $x^3 - 2x^2 - 17x - 42 = (x-6)(x^2+4x+7)$ implies that $P(6) = 0$, so that $2 \cdot 1296 + 216b + 36c - 397 \cdot 6 + 438 = 0$ and thus that

$$6b + c = -18.$$

If Adi was lying, then Ron and Leo were telling the truth, so that $5b + c = 12$ and $6b+c = -18$. This implies that $(b, c) = (-30, 162)$. But since $(-30)^2 - 4 \cdot 162 = 252$, so Adi was also telling the truth, a contradiction.

Hence Adi was telling the truth. If Ron was lying and Leo was telling the truth, then $b^2 - 4c = 252$ and $6b + c = -18$, giving $(b, c) = (6, -54)$. If Leo was instead lying and Ron was telling the truth, then instead $b^2 - 4c = 252$ and $5b + c = 12$, giving $(b, c) = (10, -38)$. Hence the structure for P can be restored to either

$$P(x) = 2x^4 + 6x^3 - 54x^2 - 397x + 438$$

or

$$P(x) = 2x^4 + 10x^3 - 38x^2 - 397x + 438.$$

In either case we may compute that $P(-4) = \boxed{1290}$.

6. Let a, b, c, and d be positive real numbers with $(a+c)(b+d) = 1$. Prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{a+b+d} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

SOLUTION. *(Source: 1990 ISL)* Using Titu's Lemma, we have that

$$\sum_{\text{cyc}} \frac{a^3}{b+c+d} = \sum_{\text{cyc}} \frac{(a^2)^2}{a(b+c+d)} \geq \frac{\left(\sum_{\text{cyc}} a^2\right)^2}{\sum_{\text{cyc}} a(b+c+d)}.$$

By Rearrangement we have $\sum_{\text{cyc}} a^2 \geq \sum_{\text{cyc}} ab = 1$ and, so

$$3\left(\sum_{\text{cyc}} a^2\right)^2 \geq 3\left(\sum_{\text{cyc}} a^2\right) \geq \sum_{\text{cyc}} a(b+c+d).$$

This is enough to finish.

We leave a solution using Jensen as an exercise to the energetic reader.

7. A convex n-gon is contained in a unit square. Prove that there are three distinct vertices of the n-gon which determine a triangle of area less than $\frac{8}{n^2}$.

SOLUTION. *(Source: Gabriel Dospinescu)* Number the sides of our polygon from 1 to n going counterclockwise, and let a_i be the length of the side numbered i. Then

$$\sum_{i=1}^{n}(a_i + a_{i+1}) = 2\sum_{i=1}^{n} a_i \leq 8,$$

where $a_{n+1} = a_1$, so there exists some a_i such that $a_i + a_{i+1} \leq \frac{8}{n}$. Since a_i and a_{i+1} are lengths of adjacent segment of our n-gon, we can assign vertices A, B, and C such that AB is the segment with length a_i and BC is that with length a_{i+1}. Then, letting $\theta = \angle ABC$, by AM-GM we obtain

$$[ABC] = \frac{AB \cdot BC \cdot \sin\theta}{2} \leq \frac{AB \cdot BC}{2} \leq \frac{\left(\frac{AB+BC}{2}\right)^2}{2} = \frac{\left(\frac{4}{n}\right)^2}{2} = \frac{8}{n^2}.$$

It's been a great 27 years.