Berkeley Math Circle: Monthly Contest 6 Solutions

1. How many distinct products can be formed by multiplying together one or more different elements of the set $\{1, 2, 3, 4, 8\}$?

   For example, the product $2 \cdot 3$ is equivalent to the product $1 \cdot 2 \cdot 3$ and so should both only be counted as one distinct product, not two. Equivalently, the ordering of the multiplication should not be considered. For instance, the products $1 \cdot 2 \cdot 3$ and $2 \cdot 3 \cdot 1$ are equivalent.

   SOLUTION. The possible products formed must be a factor of $1 \cdot 2 \cdot 3 \cdot 4 \cdot 8 = 2^6 \cdot 3$. Since $\{1, 2, 4, 8\} = \{2^0, 2^1, 2^2, 2^3\}$, all values of $2^i 3^j$ for $0 \leq i \leq 6$ and $0 \leq j \leq 1$ can be attained. Hence all factors of $2^6 \cdot 3$ can be attained, and the answer is $7 \cdot 2 = \boxed{14}$.

2. Let $ABCD$ be a convex quadrilateral with $AD = 25$, $AB = 7$, $BD = 24$, $AC = 20$, and $CD = 15$. Let $EFGH$ be an isosceles trapezoid formed by connecting three equilateral triangles of side length $\frac{25}{\sqrt{3}}$. Which has larger area, quadrilateral $ABCD$ or quadrilateral $EFGH$?

   SOLUTION. Since $AB^2 + BD^2 = AC^2 + CD^2 = AD^2$, by the Pythagorean Theorem we have $\angle ABD = \angle ACD = 90°$. Hence $ABCD$ is cyclic with circumradius $\frac{25}{2}$, so it can be inscribed within a semicircle of diameter $AD = 25$.

   Now note that an equilateral triangle of side length $\frac{25}{\sqrt{3}}$ has an altitude of length $\frac{25}{2}$. We can thus inscribe a sector of a circle with radius $\frac{25}{2}$ and subtending an angle of $60°$ within the equilateral triangle. This implies that $EFGH$ can circumscribe a semicircle of diameter 25. As a result, quadrilateral $\boxed{EFGH}$ can completely contain quadrilateral $ABCD$ and therefore has larger area.

3. A stack of n cards is labeled with the integers from 1 to n such that the kth bottommost card is labeled with k. Thus the topmost card is labeled with n, the second topmost card is labeled with $n - 1$, all the way to the bottom card labeled with 1.

   Aerith has two boxes, a red box and a blue box. At all points in time, the *score* of a box is the sum of the numbers written on the cards it contains. Every minute, Aerith draws the topmost card from the stack and inserts it in whichever box currently has the lower score, breaking ties arbitrarily. Let R and B respectively be the scores of the red and blue boxes at the very end. Prove that $R - B$ is either -1, 0, or 1.

   SOLUTION. We proceed by strong induction. By simulating the problem, we find that for the cases of $n = 1$, $n = 2$, $n = 3$, and $n = 4$ the final values of the unordered pair $\{R, B\}$ are respectively $\{0, 1\}$, $\{1, 2\}$, $\{3, 3\}$ and $\{5, 5\}$, which all work.

   Now let $n \geq 5$ suppose that the desired statement holds for all $n' < n$. In the first four steps, observe that Aerith will put the cards labeled with n and $n - 3$ in one box, and the cards labeled with $n - 1$ and $n - 2$ in another. At this point the scores of the two boxes are the same, and we finish by the inductive hypothesis on $n - 4$.

4. For any integer k, let $\rho(k)$ denote the set of prime numbers that divide k. As an example, note that $\rho(35) = \{5, 7\}$ and $\rho(36) = \{2, 3\}$.

   Determine whether or not there exist infinitely many pairs of positive integers (m, n) that simultaneously satisfy the equalities $\rho(m) = \rho(n)$ and $\rho(m+1) = \rho(n+1)$.

   SOLUTION. The answer is $\boxed{\text{yes}}$. For instance, take $(m, n) = (2^k - 2, (2^k - 1)^2 - 1)$ for any $k \geq 2$. Observe that $n = m(m+2)$. Since $m+2 = 2^k$ and $\rho(ab) = \rho(a) \cup \rho(b)$ for all a and b, we have $\rho(n) = \rho(m) \cup \rho(m+2) = \rho(m) \cup \{2\} = \rho(m)$. But $n + 1 = (m+1)^2$, so it is immediate that $\rho(n+1) = \rho(m+1)$.

5. Let n be a positive integer. What is the value of

$$\sum_{k=0}^{n-1} (-1)^k k^{n-1} \binom{n}{k}?$$

   SOLUTION. The answer is $\boxed{(-n)^{n-1}}$. Equivalently, it suffices to show that

$$\sum_{k=0}^{n} (-1)^k k^{n-1} \binom{n}{k} = 0.$$

   But the above summation counts using inclusion and exclusion exactly the number of ways to form a string of length n from an alphabet of $n-1$ characters such that all letters are pairwise distinct, which is exactly 0. We are done.

6. Does there exist a two-player game, played only with a fair coin, such that

   - the probability that the game will never terminate is zero, and
   - the probability of one of the players winning is an irrational number?

   SOLUTION. *(Source: Putnam)* The answer is $\boxed{\text{yes}}$. Let Aerith and Bob be the two players.

   Here is the construction. Divide the game into a sequence of rounds, starting with the first. On the nth round, let the game end in a win for Aerith with probability $\frac{1}{n+1}$ and let the game end in a win for Bob with probability $\frac{n-1}{n+1}$. Otherwise, let the game progress to the next round; this occurs with probability $\frac{1}{n+1}$. This is attainable using a fair coin as it can simulate any collection of rational probabilities of sum 1.

   Clearly, the probability that the game never ends is zero, and Aerith wins on the jth round with probability $\prod_{j=1}^{n} \frac{1}{j+1}$, as the last round must end in a win for Aerith and all other rounds were draws. Hence Aerith wins with probability

$$p = \sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{1}{j+1} = \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2,$$

   which is irrational.

   Note that by changing the probabilities of $\frac{1}{n+1}$ and $\frac{n-1}{n+1}$ with other changing rational values, we can attain many other irrational values for p.

7. Fixed positive rational numbers p and q are chosen such that both $2\sqrt{p}$ and $q - p$ are integers. Let $\alpha = \sqrt{p} + \sqrt{q}$, and define the set S such that it contains exactly all numbers of the form $m\alpha + n$, where m and n are any integers. Given that α is not an integer, show that the set $S \cap (0, r)$ is nonempty for all positive real numbers r.

   SOLUTION. Letting $k = 2\sqrt{p}$ and $l = q - p$ be integers, note that $\alpha^2 = k\alpha + l \in S$. Going inductively, every perfect power of α therefore lies in S.

   Since $\alpha \notin \mathbb{Z}$, there exists $d \in \mathbb{Z}$ such that $\alpha + d \in (0, 1)$. Hence, there exists some $n \in \mathbb{Z}$ such that $(\alpha + d)^n \in (0, r)$, but via binomial expansion we find that $(\alpha + d)^n \in S$ using our above observation that $\{\alpha^k \mid k \in \mathbb{Z}\} \subseteq S$. This in particular implies that $(\alpha + d)^n \in S \cap (0, r)$ as desired.