# Berkeley Math Circle: Monthly Contest 4 Solutions

1. Among all positive whole numbers from 1 to 2024, are there more numbers that are divisible by 5 but not by 6, or are there more numbers that are divisible by 6 but not by 5?

   **SOLUTION.** Let $A$ be the set of all integers between 1 and 2024 that are divisible by 5 but not by 6. Let $B$ be the set of all integers between 1 and 2024 that are divisible by 6 but not by 5. Let $C$ be the set of all integers between 1 and 2024 that are divisible by 30. Observe that $A$, $B$, and $C$ are mutually disjoint. Additionally, $A \cup C$ is the set of all integers between 1 and 2024 divisible by 5, while $B \cup C$ is the set of all integers between 1 and 2024 divisible by 6. Hence $A \cup C$ has more elements than $B \cup C$, which means that $A$ has more elements than $B$. The answer is thus that there are more $\boxed{\text{numbers divisible by 5 but not by 6}}$.

2. There is a spinner, divided into $n$ unequal sectors, each labeled with the numbers 1 through $n$. For all integer values of $k$ with $1 \leq k \leq n$, the spinner lands on the number $k$ with a probability that is exactly $k^2$ times the probability that it lands on the number 1. Also, it turns out that there exists some number $m$ such that the probability of the spinner landing on a number less than $m$ is exactly $\frac{1}{11}$. What is the smallest possible value of $n$?

   **SOLUTION.** Set $a_x = 1^2 + 2^2 + \cdots + x^2$ for all integers $x$. The probability that the spinner lands on $k$ is $\frac{k^2}{a_n}$, and thus the probability that the spinner lands on some number less than $m$ is $\frac{1^2}{a_n} + \frac{2^2}{a_n} + \cdots + \frac{(m-1)^2}{a_n} = \frac{a_{m-1}}{a_n}$. This probability is also $\frac{1}{11}$, so $a_n = 11a_{m-1}$. Observing that $(a_1, a_2, a_3, a_4, a_5) = (1, 5, 14, 30, 55)$, with $\frac{a_2}{a_5} = \frac{1}{11}$, by inspection the smallest possible value of $n$ is $\boxed{5}$, for which $m = 3$.

3. Let $ABCDE$ be a regular square pyramid with square base $ABCD$ such that $AB = 1$ and $AE = \frac{\sqrt{5}}{2}$. Let $F$ be the midpoint of $AB$ and let $G$ be the center of the square $ABCD$. Let $H$ lie on $EG$ such that $FH$ bisects $\angle EFG$. Find $GH$.

   **SOLUTION.** Since $ABCDE$ is regular, the triangle $\triangle ABE$ must be isosceles, in which $EF$ is an altitude. Thus $\triangle AFE$ is a right triangle with legs $AF$ and $EF$. By the Pythagorean Theorem, it follows that $EF = \sqrt{AE^2 - AF^2} = \sqrt{\frac{5}{4} - \frac{1}{4}} = 1$.

   Let $I$ be the midpoint of $CD$, and note that $F$, $G$, and $I$ are collinear. Then note that $EI = FI = 1$, so $\triangle EFI$ is equilateral, giving $\angle EFI = \angle EFG = 60°$. In particular, $\angle GFH = 30°$. Additionally, observe that $\angle FGH = 90°$ and $FG = \frac{1}{2}$ as $G$ is the midpoint of $FI$. Thus $\triangle FGH$ is a 30°-60°-90° triangle, giving $GH = \frac{FG}{\sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}}}$.

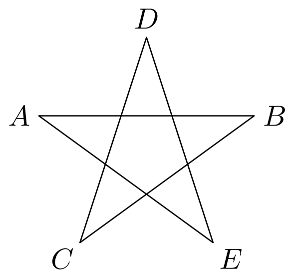
4. Let $n \geq 3$ be a positive integer. Consider polynomials of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x$ for which none of $a_n$, $a_{n-1}$, and $a_1$ are integers. Determine, with proof, whether it is possible for $P(m)$ to be an integer whenever $m$ is an integer.

**SOLUTION.** The answer is $\boxed{\text{yes}}$. Consider the polynomial function

$$P = \frac{x(x+1)\cdots(x+n-1)}{n!}.$$

One can compute that $a_n = \frac{1}{n!}$, $a_{n-1} = \frac{1}{2(n-2)!}$, and $a_1 = \frac{1}{n}$, and these are can never take on integer values when $n \geq 3$. However, note also that $P = \binom{x+n-1}{n}$, so $P(m)$ is an integer whenever $m$ is.

5. Observe that the five-pointed star is a self-intersecting pentagon such that each segment crosses exactly two other segments. For instance, in the following diagram, segment $AB$ crosses segments $CD$ and $DE$.




Is it possible to draw a self-intersecting pentagon such that each segment crosses exactly one other segment? What about a self-intersecting 20021-gon?

**SOLUTION.** The answer is $\boxed{\text{no}}$ in both cases. It suffices to demonstrate the above result where 5 and 20021 are replaced by some odd $n$.

If each segment of the $n$-gon crosses one and only one other segment, we remove any two segments crossing each other. This does not affect the other crossing points, because the two removed segments do not cross any of the remaining $n-2$ segments. Of the remaining $n-2$ segments, choose any intersecting couple and remove them. After repeating this procedure sufficiently many times, we will end up with a single remaining segment that has nothing left to intersect with, a contradiction. Therefore, it is impossible to construct a self-intersecting $n$-gon such that each segment crosses one and only one other segment.

6. A function $a(x, y)$ is said to be an *increasing integer partition* if, for all positive integers $k$, the sequence $(a(k,1), a(k,2), a(k,3), \dots)$ is increasing, and if, for all positive integers $n$, there exists exactly one pair of positive integers $(i, j)$ such that $a(i, j) = n$. Is it possible that there exists some increasing integer partition $a(x, y)$ for which $a(x, y) \leq f(x+y)$, for all positive integers $x$ and $y$, where
   (a) $f(x) = x^2$?
   (b) $f(x) = Cx^{1.5}$ for some constant $C$?
   (c) $f(x) = Cx^{1.01}$ for some constant $C$?

**SOLUTION.** The answers are $\boxed{\text{(a) yes, (b) no, and (c) no}}$. We first begin with the impossible cases.

As $f(x)$ is increasing, we have that $a(i,j) \le Cf(i+j) \le Cf(x)$ whenever $i+j \le x$ for any given $i$, $j$, and $x$. Since the values of $a(i,j)$ partition the positive integers, every $a(i,j)$ such that $i+j \le x$ must be a different positive integer less than or equal to $Cf(x)$. There are exactly $\binom{x}{2} = \frac{x(x-1)}{2}$ such pairs of positive integers $(i,j)$ such that $i+j \le x$, so we must have $x(x-1) \le 2Cf(x)$ for all $x$. Hence, if $f(x)$ exhibits $O(x^b)$ growth, then necessarily $b > 2$, establishing that (b) and (c) both fail.

For the case where $f(x) = x^2$, we will provide a construction as follows. First, we number the lattice points in the first quadrant of the $xy$-plane. Start by numbering $(1,1)$ as 1. If the last numbered point is $(a,b)$, put the next integer at $(1, a+1)$ if $b = 1$ and at $(a+1, b-1)$ otherwise. Since this numbers each down-diagonal in the positive integer lattice, every lattice point eventually will be assigned a number.

Then, set $a(i,j)$ to be the number on the lattice at $(i,j)$. The positive integers are partitioned by the sequences because each positive integer is assigned to exactly one lattice point. Also, each sequence is increasing because when $j < k, (i,j)$ will be numbered before $(i,k)$, so $a(i,j) < a(i,k)$ when $j < k$ and the sequences are increasing. Finally, the point $(i,j)$ will be numbered before any points outside of the square with boundary points $(1,1), (i+j,1), (1,i+j)$, and $(i+j,i+j)$ are numbered. Therefore, the number at the point $(i,j)$ must be less than or equal to $(i+j)^2$, so each $a_{i_j} \le (i+j)^2 = f(i+j)$ and we are done.

7. Solve over the real numbers the system of equations

$$\begin{aligned} ca(a-8) &= a(3a-59) + (2-1)(83-16c), \\ ab(b-12) &= b(b-33) + (3-1)(39-18a), \\ bc(c-10) &= c(2c-35) + (2+3)(19-5b). \end{aligned}$$

**SOLUTION.** The given system rearranges to

$$\begin{aligned} (a-4)^2(c-3) &= -35(a-1), \\ (b-6)^2(a-1) &= -21(b-2), \\ (c-5)^2(b-2) &= -15(c-3). \end{aligned}$$

Then clearly the only possible solution to our system is $(a,b,c) = \boxed{(1,2,3)}$, which works.