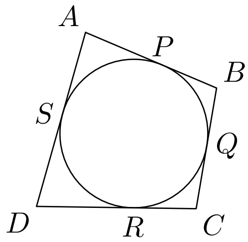
Berkeley Math Circle: Monthly Contest 3 Solutions

1. Let $ABCD$ be a quadrilateral. There is a circle tangent to the sides AB, BC, CD, and DA, as shown. Show that $AB + CD = AD + BC$.



SOLUTION. Let P, Q, R, and S respectively be the points of tangencies between the incircle ω and the segments AB, BC, CD, and DA. Then $AP = AS$, $BP = BQ$, $CQ = CR$, and $DR = DS$, so that $AB + CD = AP + BP + CR + DR = AS + BQ + CQ + DS = AD + BC$, as desired.

2. The infinite city of Metropolis is built on a rectangular coordinate system. For all whole numbers x and y, the Metropolians have built a building of height $xy - 6x - 4y$ at the location (x, y). (Buildings of negative height are built as underground cellars.) How many buildings have a height of 2024?

SOLUTION. Suppose that the building at (x, y) has height 2024, so that $(x-4)(y-6) = xy-6x-4y+24 = 2048 = 2^{11}$. Hence, we have either $(x, y) = (2^a+4, 2^{11-a}+6)$ or $(x, y) = (-2^a + 4, -2^{11-a} + 6)$ for some integer $0 \le a \le 11$. It now suffices to determine when x and y are both nonnegative. If $(x, y) = (2^a + 4, 2^{11-a} + 6)$, we always have $x > 0$ and $y > 0$. If $(x, y) = (-2^a + 4, -2^{11-a} + 6)$ and x and y are both nonnegative, then $2^a \le 4$ and $2^{11-a} < 6$, so that $2^{11} < 24$, a contradiction. Hence our solution set is exactly the pairs of the form $(x, y) = (2^a + 4, 2^{11-a} + 6)$ where the integer a satisfies $0 \le a \le 11$, giving 12 options for (x, y) and thus $\boxed{12}$ buildings.

3. For which integers n is $(n^2 + n)^2 + 3$ a prime number?

SOLUTION. Observe that $(n^2 + n)^2 + 3 = (n^2 - n + 1)(n^2 + 3n + 3)$. If this is prime, then at least one of these factors has absolute value 1. In other words, either $n^2 - n + 1 = 1$, $n^2 - n + 1 = -1$, $n^2 + 3n + 3 = 1$, or $n^2 - n + 1 = -1$. Solving for n, we find that $n \in \boxed{\{-2, -1, 0, 1\}}$, and all four of these values work.

4. A positive integer n is said to be *magnificent* if, for some $k > 1$, it is possible to write n as the product of k positive integers as well as the sum of the exact same k positive integers. For example, note that 6 is magnificent as $6 = 3 + 2 + 1 = 3 \cdot 2 \cdot 1$. This construction uses $n = 6$ and $k = 3$.

Is 99 a magnificent number? What about 101?

SOLUTION. First we show that any prime number p is not magnificent. This is because the only positive integer factors of p are 1 and p, so any sequence of k terms multiplying to p must include exactly one p. But then the sum of these terms must strictly exceed p, as $k > 1$.

Since 101 is prime, it follows that it is not magnificent.

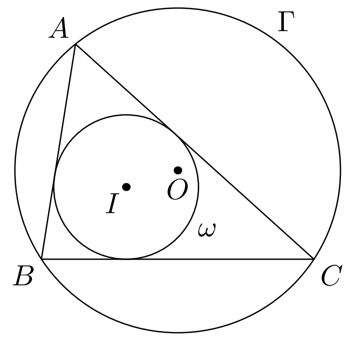
On the other hand, note that 99 is magnificent as

$$99 = 33 \cdot 3 \cdot \underbrace{1 \cdot \dots \cdot 1}_{63 \text{ times}} = 33 + 3 + \underbrace{1 + \dots + 1}_{63 \text{ times}}.$$

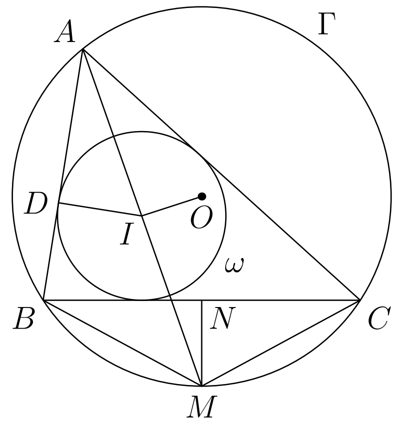
In fact, one can show that all composite numbers are magnificent.

Hence $\boxed{\text{99 is magnificent, while 101 is not}}$.

5. Circles Γ and ω are drawn in the plane with ω wholly contained inside of Γ. Distinct points A, B, and C are chosen on Γ such that the segments AB, BC, and CA are all tangent to ω, as shown. Let Γ have center O and radius R, and let ω have center I and radius r. Prove that $OI^2 + 2Rr = R^2$.



SOLUTION.



Let D be the tangency point between the circumcircle ω of $\triangle ABC$ and AB, and let M and N respectively be the midpoints of arc BC and segment BC, with arc BC chosen so that AM and BC intersect. Setting $\alpha = \frac{\angle BAC}{2}$, we observe the angle equalities $\angle BAM = \angle MAC = \angle NBM = \angle NCM = \frac{\alpha}{2}$. Then $AI = \frac{DI}{\sin \angle DAI} = \frac{r}{\sin \alpha}$ and $IM = BM = \frac{BN}{\cos \angle NBM} = \frac{BC}{2 \cos \alpha}$. Then by Power of a Point we finish as

$$R^2 - OI^2 = \text{Pow}_\omega(I) = AI \cdot IM = \frac{r \cdot BC}{2 \sin \alpha \cos \alpha} = \frac{r \cdot BC}{\angle BAC} = 2Rr.$$

6. A small company that provides medical services has an ambulance and five employees, Aerith, Bob, Cantor, Perelman and Landau. When they receive a call for services, they send the ambulance with a team of three employees. It turned out that at the end of a day Aerith made five trips, which is more than anyone else, while Bob made two trips, which is fewer than anyone else. How many trips did the ambulance make on that day?

SOLUTION. Let A, B, C, P, and L be the number of trips made by Aerith, Bob, Cantor, Perelman, and Landau, respectively. Since C, P, L are all either 3 or 4, it follows that $16 = 5 + 2 + 3 + 3 + 3 \leq A + B + C + P + L \leq 5 + 2 + 4 + 4 + 4 = 19$. However, the number of trips made by the ambulance is exactly $n = \frac{A+B+C+P+L}{3}$ and is an integer. This forces the equality $A + B + C + P + L = 18$ and $n = \boxed{6}$.

7. What is the value of

$$\sum_{k=1}^{365} \frac{365! \cdot k}{(365-k)! \cdot 365^k}?$$

SOLUTION. Consider a year with 365 days. One by one we select birthdays, among all 365 days uniformly at random, stopping once some birthday has been selected twice. This process must terminate by or on the 366th selection. The probability that k total birthdays have been selected by when the process ends is

$$\left(\frac{365!}{(365-k)! \cdot 365^k}\right)\left(\frac{k}{365}\right),$$

where the first term is the probability that the first k birthdays selected are all pairwise distinct, and the second term is the probability that the $(k+1)$th birthday selected is one of the first k. Thus

$$\sum_{k=1}^{365}\left(\frac{365!}{(365-k)! \cdot 365^k}\right)\left(\frac{k}{365}\right) = 1,$$

so the answer is $\boxed{365}$.