Berkeley Math Circle: Monthly Contest 7 Solutions

1. The *factorial* of a positive integer n, denoted $n!$, is the product of all positive whole numbers less than or equal to n. For example, $7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$.

   The *double-factorial* of a positive integer n, denoted $n!!$, is the product of all *even* positive whole numbers less than or equal to n. For instance, $7!! = 6!! = 2 \cdot 4 \cdot 6$.

   Which is greater, $(2024!!)!$ or $(2024!)!!$?

   SOLUTION. For any even n, we compare terms to find that

$$\begin{aligned}(n!!)^2 &= (n)(n) \cdot (n-2)(n-2) \cdot (n-4)(n-4) \cdots (2)(2) \\ &> n(n-1) \cdot (n-2)(n-3) \cdot (n-4)(n-5) \cdots (2)(1) = n!.\end{aligned}$$

   A similar comparison yields $(2n)! > (n!)^2$. Putting these together yields

$$((2024!)!!)^2 > (2024!)! > \left(\left(\frac{2024!}{2}\right)!\right)^2 > ((2024!!)!)^2,$$

   so $(2024!)!! > (2024!!)!$. Thus $\boxed{(2024!)!!}$ is greater.

2. There is a curvy loop in the plane, with no straight line intersecting it at more than two points. Prove that any point on the curvy loop is a vertex of an equilateral triangle whose vertices all lie on the curvy loop.

   SOLUTION. Let us fix a point P on the loop and a very small angle θ. Now, starting from P, imagine facing inwards into the loop and then looking left along the line tangent to the loop at P.

   Now, rotate your view rightwards by θ and measure the distance, denoted d_1, to the point on the loop in that direction. After this, rotate right by another $60°$ and measure the distance, denoted d_2, in that direction.

   Observe that if θ is sufficiently small, smaller than some threshold $\theta_1 > 0°$, then we will have $d_1 < d_2$. Conversely, if θ is sufficiently close $120°$, larger than some threshold $\theta_2 < 120°$, we will then have $d_2 > d_1$.

   Hence, if we were to start at $\theta = \theta_1$ in which $d_1 < d_2$, and slowly increase θ to θ_2 in which $d_1 > d_2$, the above guarantees some value of θ such that $d_1 = d_2$, since d_1 and d_2 change smoothly without any jumps. This implies the existence of an equilateral triangle because one angle is $60°$ and the adjacent sides have equal length.

3. Find all functions f from the integers to the integers such that there exists another function g, also from the integers to the integers, such that $g(0) \neq 0$ and

$$f(xg(y) + z) = f(yg(x) + z)$$

   for any integers x, y, and z.

SOLUTION. We show that f can be $\boxed{\text{any periodic function}}$.

First, we show that any periodic function works for f. Indeed, if f is periodic with positive period a, let g be the constant function $g(x) = a > 0$. For any integers x, y, and z, we have $f(xg(y) + z) = f(ax + z)$ and $f(yg(x) + z) = f(ay + z)$. Since x and y are both integers, periodicity yields $f(z) = f(ax + z) = f(ay + z)$ as desired.

It now suffices to show that any function f satisfying the given conditions is. Setting $g(0) = a \neq 0$, $x = 0$, and $y = 1$ implies that $f(z) = f(g(0) + z) = f(z + a)$ for all integers z. Hence f must be periodic.

4. Find all ordered triplets of integers (x, y, z) satisfying $z^2 = x^2 + xy$ and $y^2 = z^2 + xz$.

SOLUTION. Suppose that $x = 0$. This forces us to have $z^2 = 0$ and $y^2 = z^2 + z$, so $x = y = z = 0$.

If $z = 0$, we have $x^2 + xy = 0$ and $y^2 = 0$, likewise implying that $x = y = z = 0$.

Otherwise, if both x and z are nonzero, from the first given equation, we have that $y = \frac{z^2 - x^2}{x}$, from which substitution into the second equation yields

$$\frac{(z^2 - x)^2}{x^2} = z^2 + xz$$

and therefore that

$$(x - z)^2(x + z)^2 - x^2 z(x + z) = 0.$$

In particular, we define $r = \frac{x}{z}$ and homogenize the above by dividing both sides by $z^4 \neq 0$ to get

$$(r + 1)(r^3 - 2r^2 - r + 1) = (r - 1)^2(r + 1)^2 - r^2(r + 1) = 0.$$

Since x and z are both integers, it follows that either $r = -1$ or r is a rational root to the polynomial $p(r) = r^3 - 2r^2 - r + 1$. If such r exist, then by the Rational Root Theorem we have $r \in \{-1, 1\}$. One can verify that neither $p(-1)$ and $p(1)$ are zero, forcing us to have $r = -1$, so $x + z = 0$ and $y = \sqrt{z(z + x)} = 0$.

Hence the only solutions to our original system are of the form $(x, y, z) = \boxed{(k, 0, -k)}$ for any integer k.

5. Let n be some positive integer such that $2n + 5$ divides either $5n! + 2$ or $5n! - 2$. Find all possible values of n.

SOLUTION. Observe that $n = 1$ works and $n = 2$ and $n = 3$ fail. From now on, suppose that $n \geq 4$.

Let us suppose that $2n + 5$ is composite, so that there exists some factor k of n such that $1 < k < \sqrt{2n + 5}$. Since $n \geq 4$, it follows that $k \leq \sqrt{2n + 5} \leq n$, giving $k \mid n!$. Combined with the fact that k divides $2n + 5$ implies that k divides either $5n! + 2$ or $5n! - 2$, it then follows that $k = 2$. However, this is a contradiction as $2n + 5$ is odd. It therefore follows that $2n + 5$ must be prime.

By Wilson's Theorem, we then note that

$$\begin{aligned}
-400 &\equiv 400(2n+4)! \\
&\equiv 400(1\cdot 2\cdots n)(n+1)(n+2)(n+3)(n+4)((n+5)(n+6)\cdots(2n+4)) \\
&\equiv 25(n!)(2n+2)(2n+4)(2n+6)(2n+8)(-1)^n(n!) \\
&\equiv (5n!)^2(-1)^n(-3)(-1)\cdot 1\cdot 3 \\
&\equiv 9(5n!)^2(-1)^n.
\end{aligned}$$

We note that n satisfies our desired condition iff $(5n!)^2 \equiv 4 \pmod{2n+5}$ and $2n+5$ is prime. Then all inverses modulo $2n+5$ are defined, so our first condition is equivalent to having $400 \equiv -9(5n!)^2(-1)^n \equiv -36(-1)^n \pmod{2n+5}$. Hence n is satisfactory iff

$$100 \equiv -9(-1)^n \pmod{2n+5}$$

and $2n+5$ is prime.

If n is even, this implies that $2n+5$ divides 109. Since 109 is prime, this yields $n = 52$, which is even and thus valid.

If n is odd, we get $2n+5$ divides 91, so that $2n+5 \in \{7, 13, 91\}$. The condition that $2n+5$ is prime forces us to have $n = 1$, though this case was found at the very beginning, outside our our assumption that $n \geq 4$.

Either way, it follows that the solution set for n is $\boxed{\{1, 52\}}$.

6. The creatures of Mathland believe in the following creation myth.
   - In the beginning, there was nothing, not even space.
   - Then came the v *vertices*, a finite collection of points.
   - Then came the e *edges*, each created by an imaginary path connecting two edges.
   - Then came the f *faces*, each created by an imaginary film attached to a circle of edges.
   - Then came the c *cells*, each created by an imaginary solid whose boundary is a set of faces that are attached to a polyhedron of polygons.
   - A *world* is the set of all vertices that can be reached from a starting vertex by means of edges.
   - A *loop* is a (potentially empty) set of edges that uses each vertex an even number of times. Two loops are considered the same if the set of edges where they disagree forms the boundary of a set of faces.
   - A *creature* is a (potentially empty) set of faces that uses each edge an even number of times. Two creatures are considered the same if the set of faces where they disagree forms the boundary of a cell.
   - A *god* is a (potentially empty) set of cells that uses each face an even number of times.

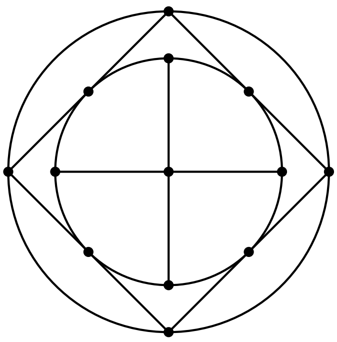
   Let κ, λ, γ, and ω respectively denote the number of creatures, loops, gods, and worlds. Prove that

$$\frac{\kappa}{\lambda\gamma} = 2^{v-e+f-c-\omega}.$$

SOLUTION. In this solution, for sets A and B, the set $A \triangle B$ contains exactly those elements that are in one set but not the other.

- In the beginning, there was nothing. The only creatures, loops, and gods were the empty set, so the formula was true then.
- Then, as each vertex was added one by one, the number of worlds ω also increased by one, which did not change the exponent on the right hand side, so the equation was still true.
- Then, as each edge was added, ω went down by one, except for when the new edge already connected two vertices in the same world, which would create a loop L. In fact, whenever we add on a new loop like this, we are actually doubling the total number of loops λ, because for each old loop L', the set $L \triangle L'$ is also a loop. Thus, the formula was still true.
- Then, as each face F was added, f increased by one. However, λ had also changed, because now some of our loops are the same. Indeed, if $b(F)$ denotes the boundary of F, then each loop L has become $L \triangle b(F)$, and so λ also decreased by a factor of two. However, maybe we already knew that L was the same as $L \triangle b(F)$ because they disagree only on the boundary of a set of faces, say F'. In this case, we have $L \triangle L \triangle b(F) = b(F')$. Note that $L \triangle L = \emptyset$ and $\emptyset \triangle b(F) = b(F)$ and so $b(F) = b(F')$. In this case, λ does not change but we do notice that we have created a new creature, namely $C = F \triangle F'$. In fact, we have doubled the total number of creatures κ because, if C' is an old creature, then $C \triangle C'$ is also a creature. Thus, the equation is still true.
- Then, as each cell X was added, c increased by one. However, κ also changed, because now some of our creatures are the same. Indeed, if $b(X)$ denotes the boundary of X, then each creature C is now the same as $C \triangle b(X)$, and so κ also decreased by a factor of two. However, maybe we already knew that C was the same as $C \triangle b(X)$ because they disagree only on the boundary of a set of cells, say X'. In this case, we have $C \triangle C \triangle b(X) = b(X')$. Note that $C \triangle C = \emptyset$ and $\emptyset \triangle b(X) = b(X)$ and so $b(X) = b(X')$. In this case, the λ does not change but we do notice that we have created a new god, namely $G = X \triangle X'$. In fact, we have doubled the total number of gods γ because, if X' is an old god, then $X \triangle X'$ is also a god. Thus, the equation is still true.

7. Aerith randomly erases some of the 24 edges in the following diagram, such that each edge is erased independently with probability $0.5 - p$ for some fixed $p \in [-0.5, 0.5]$. Prove that the expected number of bounded regions into which the plane is divided by the new figure is given by $12p + 12f(p)$, where f is an odd Lipshitz polynomial.



Note: A function g is said to be *Lipshitz* if it satisfies $|g(x) - g(y)| \leq |x - y|$ for all x and y in its domain.

SOLUTION. The figure is the graph of the diminished rhombic dodecahedron, a self-dual polyhedron. In particular, any graph with e edges, v vertices, ℓ loops, and c connected components satisfies the relation $e - v = \ell - c$. Let $c(p)$ and $\ell(p)$ denote the expected number of connected components and loops for some given value of p.

The figure that Aerith creates has an expected value of $12+24p$ edges and always has 13 vertices, which might not be connected to any edges. Each connected component corresponds to a path of adjacent faces that uses erased edges, which corresponds to a loop in the dual graph where erased edges are replaced by unerased edges. Note, however, that this requires choosing an inside for each loop, but each loop has two possibilities for choice of inside; to remedy this, simply choose the interior to be whichever component does not contain the central vertex.

Put differently, we must therefore have $c(p) - \ell(-p) = 1$, with the difference corresponds to the connected component of the central vertex. We therefore conclude that $\ell(p) - \ell(-p) = 24p$ and therefore that $\ell(p) - 12p = \ell(-p) - 12(-p)$, so $f(p) = \frac{\ell(p)}{12} - p$ is odd. Additionally, note that L is an increasing function of p and so that if $y \geq x$,

$$f(x) - f(y) = \frac{\ell(x) - \ell(y)}{12} + (y - x) \leq y - x,$$

so taking absolute values implies that f is Lipshitz.

To see that f is a polynomial, it suffices to show that L is a polynomial; for this, note that L can be computed by the principle of inclusion and exclusion to be an alternating sum of the probabilities that certain loops are included, but the probability that a given loop with e edges is included is given by $(1 - (p - \frac{1}{2}))^e$, a polynomial in e.