Berkeley Math Circle: Monthly Contest 6 Solutions

1. Is the number $23^{24} - 24^{23}$ positive or negative?

   SOLUTION. By the Binomial Theorem, we have that

$$\left(\frac{24}{23}\right)^{23} = \left(1+\frac{1}{23}\right)^{23} = 1 + \frac{\binom{23}{1}}{23} + \frac{\binom{23}{2}}{23^2} + \cdots + \frac{1}{23^{23}} \le 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{23!},$$

   where the last step is due to the fact that $\binom{23}{n} = \frac{23!}{n!(23-n)!} = \frac{23\cdot 22\cdots(24-n)}{n!} \le \frac{23^n}{n!}$ for all integers $1 \le n \le 23$. Also, clearly $(n+1)! > 2^n$ for the same n, giving

$$\left(\frac{24}{23}\right)^{23} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{23!} \le 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{22}} \le 1 + 1 + 1 = 3.$$

   Hence $\left(\frac{23}{24}\right)^{23} \ge \frac{1}{3}$, from which it follows that

$$\frac{23^{24} - 24^{23}}{24^{23}} = \frac{23 \cdot 23^{23}}{24^{23}} - 1 = 23\left(\frac{23}{24}\right)^{23} - 1 \ge \frac{23}{3} - 1 \ge 6,$$

   implying that $23^{24} - 24^{23}$ is $\boxed{\text{positive}}$.

2. V. Enhance, the CEO of Evan Corporation LLC, has a secret favorite number c, not necessarily a whole number. He also has a special number machine with a red button: when any number x is inputted into the machine and the button is pressed, it displays the value of

$$\frac{x}{2} + \frac{c}{2x}$$

   on a screen.

   Aerith begins by inputting the value $x = 1$ into the machine; every minute, she pushes the red button and reenters the value displayed on the screen into the machine. Interestingly, Aerith notices that output of the machine eventually gets closer and closer to exactly 206. Determine the value of c.

   SOLUTION. We solve this problem for 206 replaced by any general n.

   Define the function

$$f(x) = \frac{x}{2} + \frac{c}{2x} = \frac{x^2 + c}{2x},$$

   and let a_i be the number displayed on the machine immediately after Aerith presses the red button for the ith time, so that $a_0 = 1$ and $a_{i+1} = f(a_i)$ for each i.

   For positive x, we have

$$f(x) - x = \frac{x^2 + c}{2x} - x = \frac{(x^2 + c) - 2x^2}{2x} = \frac{c - x^2}{2x},$$

   implying that $x < f(x)$ iff $x < \sqrt{c}$, $x > f(x)$ iff $x > \sqrt{c}$, and $x = f(x)$ iff $x = \sqrt{c}$. Since the a_i are all positive, they must all move in the direction of $\sqrt{c}$, so convergence yields $n = \sqrt{c}$. In particular, for $n = 206$, it follows that $c = n^2 = 206^2 = \boxed{42436}$.

3. A finite collection of circles, of any radius, in the plane are shaded blue. Your task is to shade some of the circles red, while ensuring that no red circles overlap. Prove that you can always ensure that the area of the red region is at least 10% of the area of the blue region.

SOLUTION. Let S be our collection of blue circles, and let $n = |S|$ be the number of circles that S contains. We prove the desired statement by induction on n.

If $n = 1$, the conclusion is obvious. For larger n, consider the circle ω of maximal radius r and center O, and remove all circles intersecting it. If any circle ω' removed has radius s and center O', then $s \leq r$ and $t \leq s + r \leq 2r$, where $t = OO'$. In particular, the farthest point on ω' from O is a distance $t + s \leq 3r$ away from O. As a result, the area of all circles removed is at most the area of the circle with radius $3r$ concentric with ω, which is equivalently 9 times the area of ω.

Considering the subcollection T of circles not intersecting ω, our inductive hypothesis implies the existence of a subcollection R of T such that no two circles in R overlap, and the area enclosed by R is at least $\frac{1}{10}$ of the area enclosed by T. The argument in the previous paragraph additionally shows that ω has area greater than or equal to $\frac{1}{9}$ of the area of $S \backslash T$. Coloring all circles in $R \cup \{\omega\}$ red then finishes.

4. Aerith and Bob play the following game: a positive integer n is chosen, after which Aerith and Bob alternate choosing an integer between 1 and n, inclusive, that has not been chosen. They keep a running product of all numbers that have already been chosen, and the first player to make that running product a multiple of n loses. Find all initial choices of n for which Aerith wins.

SOLUTION. Aerith wins if and only if n is even or has a perfect square greater than 1 as a divisor.

If n is even, Aerith first picks $\frac{n}{2}$, so that the loser is the player forced to pick an even number. This yields two cases:

a) If n is a multiple of 4, there are an even number of odd numbers in total, with Aerith's first number $\frac{n}{2}$ being even. Hence there are an even number of odd numbers available after Aerith's turn. In particular, after each player, starting with Bob, alternates picking odd numbers, Bob will be the first one to be forced to choose an even number and thus lose.

b) Otherwise, there are an odd number of odd numbers between 1 and n inclusive, with Aerith's first number $\frac{n}{2}$ being odd, again yielding an even number of available odd numbers after Aerith's first turn, so Aerith wins once again.

If n is odd and is not squarefree, there must exist some prime p such that $p^2 \mid n$. In this case, Aerith starts by picking $\frac{n}{p}$, so the loser is the player forced to pick a multiple of p.

Since $p \mid n$, there are $n' = \frac{(p-1)n}{p}$ numbers between 1 and n, inclusive, that are not divisible by p, with n' even by parity. Noting that Aerith's first number $\frac{n}{p}$ is a multiple of p, this implies that there are still n' numbers indivisible by p after Aerith's turn. Thus, after each player, starting with Bob, alternates picking numbers not divisible by p, Bob will be the first one forced to pick a number divisible by p, so he loses.

If n is odd and squarefree, Bob's winning strategy is to pick $n-k$ after Aerith picks any integer k. Note that this is well-defined: if Aerith picks n, she immediately loses. If instead she picks some $k < n$, it is easy to see that $n-k$ must always be available and distinct from k by parity of n.

We must still show that Bob's strategy will never cause him to lose. In our case, there exist distinct odd primes $p_1, p_2, \ldots, p_k$ such that $n = \prod_{i=1}^{k} p_i$, so the game ends if and only if at least one number has been selected that is divisible by each p_i. Since $\gcd(n,k) = \gcd(n, n-k)$, the only members of $\{p_1, p_2, \ldots, p_k\}$ that divide $n-k$ also divide k. Hence, if any of Aerith's moves doesn't cause her to lose, then Bob's move immediately after will not cause him to lose either. Hence Bob can never lose before Aerith. The game must end at some point, so Aerith must lose and Bob will win.

5. Find the number of polynomials $P(x)$ of degree 3 with nonnegative integer coefficients strictly less than 100 such that the last two digits of $P(x)$ are either 00 or 76 for all integer values of x.

SOLUTION. Call a polynomial P *cool* if it satisfies our desired property above. Additionally, call a cubic Q 4-*cool* if $Q(x)$ is always divisible by 4 for any integer x, and similarly call R 25-*cool* if $Q(x)$ is equivalent to either 0 or 1 modulo 25 for the same x. Finally, for some n, say that a polynomial $S(x)$ is n-*maximal* if each of its coefficients are integers between 0 and $n-1$, inclusive. By the Chinese Remainder Theorem, note that P is cool iff it is simultaneously 100-maximal, 4-cool, and 25-cool.

For each pair of ordered quadruplets of integers (k_1, k_2, k_3, k_4) and (k_5, k_6, k_7, k_8), there is exactly one possible value of (a, b, c, d) such that $\{a, b, c, d\} \subseteq \{1, 2, \ldots, 100\}$, $(a, b, c, d) \equiv (k_1, k_2, k_3, k_4) \pmod 4$, and $(a, b, c, d) \equiv (k_5, k_6, k_7, k_8) \pmod{25}$. Thus the number of cool polynomials is equal the number of 4-maximal, 4-cool polynomials multiplied by the number of 25-maximal, 25-cool polynomials.

First, let us consider a 4-maximal and 4-cool polynomial $P(x) = ax^3 + bx^2 + cx + d$. Plugging in $x \in \{0, 1, 2, 3\}$ yield that $d = 0$ and (a, b, c) must be one of $(0, 0, 0)$, $(2, 2, 0)$, $(0, 2, 2)$, and $(2, 0, 2)$. This gives a total of 4 polynomials that are 4-maximal and 4-cool.

Now let $P(x) = ax^3 + bx^2 + cx + d$ be 25-maximal and 25-cool. Plugging in $x = 0$ yields $d \in \{0, 1\}$, and plugging in $x = 5$ grants that $\{5c, 5c + 1\} \cap \{0, 1\} \pmod{25}$ is nonempty, which forces the existence of c' with $c \equiv 5c' \pmod{25}$. In particular, it follows that $P(x) \in \{ax^3 + bx^2, ax^3 + bx^2 + 1\} \pmod 5$.

We have two cases:

a) If $d = 0$, then $P(x) \equiv ax^3 + bx^2 \pmod 5$ is either 0 or 1 modulo 5. Plugging in $x \in \{1, 2, 3, 4\}$, we can quickly conclude that $a \equiv b \equiv 0 \pmod 5$.

b) Otherwise, if $d = 1$, then $P(x) \equiv ax^3 + bx^2 + 1 \pmod 5$ is either 0 or 1 modulo 5. The same method as above also yields that $a \equiv b \equiv 0 \pmod 5$.

Hence $a \equiv b \equiv 0 \pmod 5$ either way, so we write $a \equiv 5a'$ and $b \equiv 5b' \pmod{25}$ for integers a' and b'. Since $c \equiv 5c' \pmod{25}$ and $d \in \{0, 1\}$, it therefore follows that $\{5(a'x^3 + b'x^2 + c'x^3), 5(a'x^3 + b'x^2 + c'x^3) + 1\} \cap \{0, 1\} \pmod{25}$ is nonempty is nonempty, forcing us to have $5a'x^3 + 5b'x^2 + 5c'x \equiv 0 \pmod{25}$ for all x. Thus

$a'x^3 + b'x^2 + c'x \equiv 0 \pmod 5$. Plugging in $x \in \{1, 2, 3, 4\}$ again quickly yields that $a' \equiv b' \equiv c' \equiv 0 \pmod 5$, so $a = b = c = 0$ by 25-maximality. This yields two 25-maximal and 25-cool polynomials, namely $P = 0$ and $P = 1$.

There are 4 polynomials that are 4-maximal and 4-cool and 2 polynomials that are 25-maximal and 25-cool, yielding $4 \cdot 2 = \boxed{8}$ cool polynomials in total.

6. A line is drawn in the plane. You have a straightedge, but no compass. Prove that it is impossible to construct a parallel line.

SOLUTION. As projective transformations preserve point-line incidences, taking such a transformation that preserves the original line but not its point at infinity would make the parallel line not parallel, a contradiction.

7. Let f and g be polynomials in x, y with integer coefficients.

   (a) Prove that if some integer s is expressible as a product of coefficients of f and g, there exists a positive integer n such that s^n is expressible as an integer linear combination of the coefficients of fg.

   (b) Solve part (a) where f and g are multivariable polynomials instead.

Note: an *integer linear combination* of a set of integers $\{a_1, \ldots, a_n\}$ is a number of the form $a_1b_1 + a_2b_2 + \cdots + a_nb_n$ where $\{b_1, \ldots, b_n\}$ is an arbitrary set of integers.

SOLUTION.

a) Denote f and g by

$$f(x) = f_0 + f_1x + f_2x^2 + \cdots + f_nx^n$$

and

$$g(x) = g_0 + g_1x + g_2x^2 + \cdots + g_nx^n.$$

Proceed by ascending induction on $k = i + j$ to show that $s = f_ig_j$ satisfies the desired criteria. For $k = 0$, the result is obvious because the only possibility is f_0g_0, which is a coefficient of fg.

Now assume $k \geq 1$ to prove the statement for f_ig_j. Taking convolutions yield that the kth coefficient of $h = fg$ is

$$h_k = \sum_{l=0}^{k} f_lg_{k-l},$$

so that

$$f_i^2g_j^2 = h_kf_ig_j - \sum_{l=0}^{i-1} f_lg_{k-l}f_ig_j - \sum_{l=i+1}^{k} f_lg_{k-l}f_ig_j.$$

Each term in the first summation is a multiple of f_lg_j. Since $l+j = l-i+k < k$ due to bounding, our inductive hypothesis implies that each of the terms in this summation can be raised to some power s_l to obtain a linear combination of the coefficients of fg. Analogously, each term in the second summation is a multiple of $f_i g_{k-l}$, with $i + k - l < k$ again by bounding, so our induction hypothesis tells us that each of the terms in this sum can be raised to some power s_l to obtain a linear combination of the coefficients of fg.

For the sake of notational brevity, denote $t_l = -f_l g_{k-l} f_i g_j$ for all $l \neq i$, and let $t_i = h_k f_i g_j$ so that $s_i = 1$. Our equation is now

$$(f_i g_j)^2 = \sum_{l=0}^{k} t_l.$$

If $s = \sum_{l=0}^{k} s_i$, then raising the above equation to the power of s yields a summation, with each term in the multinomial expansion of the right hand side being a multiple of $t_l^{s_l}$ for some t. Hence, the right hand side is a linear combination of the coefficients of h, completing our induction.

b) A practically identical argument follows for the $n \geq 1$ case in general. Here, we proceed by induction on n, with the $n = 1$ case following from the previous part.

For the inductive step, we suppose that $n \geq 2$. In particular, let

$$f = f_0 + \cdots + f_m$$

and

$$g = g_0 + \cdots + g_m$$

where the f_i and g_i are both homogeneous polynomials in the n variables, with a product of coefficients of f and g corresponding to a product of coefficients of f_i and g_j for some i and j. The crucial observation is that the coefficients of f_i and g_j are exactly the coefficients of $f_i(x_1, x_2, \ldots, x_{n-1}, 1)$ and $g_j(x_1, x_2, \ldots, x_{n-1}, 1)$, which are polynomials only in $n - 1$ variables.

The inductive hypothesis implies that a product of coefficients of the polynomials $f_i' = f_i(x_1, x_2, \ldots, x_{n-1}, 1)$ and $g_j' = g_j(x_1, x_2, \ldots, x_{n-1}, 1)$ can be raised to some power to get a linear combination of coefficients of $f_i' g_j'$, which are exactly just the coefficients of $f_i g_j$. We now notice that it suffices to show that $f_i g_j$ can itself be raised to a power m to get a linear combination of the homogeneous components of fg, as any coefficient c corresponding to $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ will show itself as $c^m x_1^{ma_1} x_2^{ma_2} \cdots x_n^{ma_n}$ in the expansion of $(f_i g_j)^m$. Then the argument is entirely analogous to that of the $n = 1$, where we use induction on $k = i + j$ to show our desired statement for $f_i g_j$, which finishes.