Berkeley Math Circle: Monthly Contest 5 Solutions

1. Define the function f on the positive integers such that

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even,} \\ 5x+1 & \text{if } x \text{ is odd.} \end{cases}$$

Find the smallest positive integer n for which there does not exist some positive integer m such that $f^m(n) = 1$. (In other words, we want the smallest n such that $f(n) \neq 1$, $f(f(n)) \neq 1$, $f(f(f(n))) \neq 1$, $f(f(f(f(n)))) \neq 1$, and so on.)

SOLUTION. We have $1 = 1$, $f(2) = 1$, $f^5(3) = f^4(16) = f^3(8) = f^2(4) = f(2) = 1$, and $f^2(4) = f(2) = 1$. Also, note that $f^2(5) = f(26) = 13$, and

$$f^{10}(13) = f^9(66) = f^8(33) = f^7(166) = f^6(83) = f^5(416) = f^5(32 \cdot 13) = 13.$$

Hence the values of $f^i(5)$ for $i \geq 3$ cycle among $\{13, 66, 33, 166, 83, 416, 208, 104, 52, 26\}$, implying that there is no m such that $f^m(n) = 1$. Thus our answer is $\boxed{5}$.

2. Completely factor $N = 2^{30} - 1$ into prime numbers.

SOLUTION. Using the factorizations $x^2 - 1 = (x-1)(x+1)$, $x^3 - 1 = (x-1)(x^2 + x + 1)$, and $x^3 + 1 = (x+1)(x^2 - x + 1)$, we find that

$$N = 2^{30} - 1 = (2^{15}-1)(2^{15}+1) = (2^5-1)(2^{10}+2^5+1)(2^5+1)(2^{10}-2^5+1) = 31 \cdot 1057 \cdot 33 \cdot 993.$$

By inspection, we note that $33 = 3 \cdot 11$, $1057 = 7 \cdot 151$, and $993 = 3 \cdot 331$, with both 151 and 331 being prime numbers, so our final factorization is

$$N = 31 \cdot (7 \cdot 151) \cdot (3 \cdot 11) \cdot (3 \cdot 331) = \boxed{3^2 \cdot 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331}.$$

3. Jessica owns four pairs of socks, which come in four different colors: ultramarine, vermilion, wisteria, and xanthous. Each Wednesday, she does her laundry, and every time exactly one sock goes missing.

   (a) List all possible color combinations for her four pairs of socks. Here, two color combinations are considered the same if the number of pairs of socks of the same color match; for example, having three pairs of ultramarine socks and one pair of vermilion socks and having three pairs of vermilion socks and one pair of wisteria socks are considered the same.

   (b) For each of the color combinations above, calculate the time, in weeks, until she runs out of socks to wear. Assume that she wears only matching socks at any given time, so that if she has just one sock of a given color, she throws that useless sock away.

SOLUTION.

(a) Observe that Jessica's options amounts to listing the ways to split 4 into distinct groups of addends, as given below.

   i. (4)
   ii. $(3, 1)$
   iii. $(2, 2)$
   iv. $(2, 1, 1)$
   v. $(1, 1, 1, 1)$

   Note that we list all available options without permutations. For example, splitting the color combinations between $(3, 1)$ and $(1, 3)$ are considered the same, because they do not affect the time until she runs out of socks.

   In particular, these correspond to the possible color combinations for Jessica's four pair of socks, namely, as enumerated below.

   i. $\boxed{\text{4 pairs of the same color}}$
   ii. $\boxed{\text{3 pairs of one color and 1 pair of another color}}$
   iii. $\boxed{\text{2 pairs of one color and 2 pairs of another color}}$
   iv. $\boxed{\text{2 pairs of one color, 1 pair of another color, and 1 pair of yet another color}}$
   v. $\boxed{\text{All four pairs are of different colors}}$, so that we have 1 pair of one color, 1 pair of another color, 1 pair of yet another color, and 1 pair of a final different color

(b) Now we go through each of the five options, enumerated in the above order, to determine how many weeks it will take for Jessica to run out of socks to wear.

   Consider option 1. This option means that all 4 pairs have the same color and, therefore, they are all interchangeable. Since one sock gets lost once a week, her total supply will still be wearable until it gets down to one sock, which will take $\boxed{7}$ weeks.

   Consider option 2. View this case as a case with two pools of socks, consisting respectively of 6 socks and 2 socks. Jessica will run out of socks when both pools are down to one sock, regardless of which pool is depleted first. The first pool alone can provide Jessica with 5 weeks of wearing, and the second with 1 week. The lost socks from both pools can be interspersed, which does not affect the total duration since it lasts until both pools are exhausted. Therefore, the answer is $1 + 5 = \boxed{6}$ weeks.

   Consider option 3. This case is similar to the previous one. Now we have two pools of socks which consist of 4 socks and 4 socks, respectively. Each pool alone can provide Jessica with 3 weeks of wear and the exact order in which the socks get lost again does not matter. The answer is $3 + 3 = \boxed{6}$ weeks.

   Consider option 4. By the same logic as above, now we have 3 pools of socks, each of which alone can last for 3, 1 and 1 weeks, respectively. This implies that the answer is $3 + 1 + 1 = \boxed{5}$ weeks.

   Consider option 5. This option means that each pair has its unique color and the loss of each sock results in losing the whole pair. Since each pair lasts for just one week, the entire set will last for $\boxed{4}$ weeks.

4. Six mathematicians stand around a tree in a circle. Each has a hat whose color is randomly either red or blue. They cannot see the color of their own hat, nor the color of the hat of the person across from them, but they can see the hats of the four other mathematicians. If they can pick their strategy beforehand, what is the maximum chance they could have for everyone silently guessing their hat color right?

   SOLUTION. The chances that two mathematicians directly across from each other guess right are independent, with each individual probability being $\frac{1}{2}$ because they cannot see or control the colors of their own hats. Thus the answer is at most $(\frac{1}{2})^2 = \frac{1}{4}$. We show that this is possible.

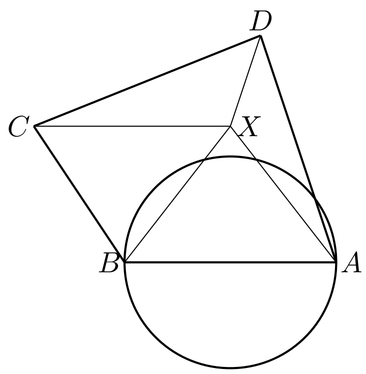
   Split the mathematicians, based on position, into two equilateral triangles. Their strategy will be to guess based off of the assumption that both triangles have an odd number of red hats and deduce the color of their own hat from those in their triangle. The chance they win is then the probability that this assumption is true, which occurs with probability

$$\left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{4}}.$$

5. Fix some positive integer n, and consider all convex polygons P with n sides. For each such P, draw n circles, with each side being a diameter. Prove that there exists some polygon P and point X in the interior of P, with X uncovered by the n circles, iff $n \geq 5$.

   SOLUTION. Call an n-gon P *coverable* if it is fully covered by the n circles with diameter given by the sides P. We say that some n is *coverable* iff all n-gons are coverable. It suffices to prove that the only coverable integers are 3 and 4.

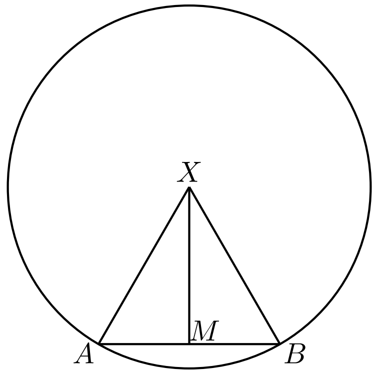
   We first prove that all quadrilaterals are coverable. By contradiction, assume that there is a point X inside a quadrilateral $ABCD$ which none of the four circles covers. In particular, since X is outside the circle with diameter AB, our exterior point condition yields that $\angle AXB < 90°$. Analogous logic similarly implies that $\angle BXC < 90°$, $\angle CXD < 90°$, and $\angle DXA < 90°$, from which summing yields $\angle AXB + \angle BXC + \angle CXD + \angle DXA < 360°$, a contradiction.



   One can check that all triangles are coverable via a directly analogous argument. In particular, the above arguments imply that 3 and 4 are both coverable.

Now we show that any $n \geq 5$ is not coverable by showing that such a regular n-gon is not coverable.

In particular, draw a circle centered at X and inscribe a regular polygon with $n > 4$ in that circle. Let AB be a side of the polygon with midpoint M. As $n \geq 5$, we have $\angle AXB < 90°$ and thus $\angle AXM < 45°$. Hence $AM < XM$, so the circle constructed on AB as a diameter does not reach X. The same applies to all n sides of the regular polygon, which shows that the regular n-gon is not coverable with respect to X.



As an extra remark, note that it does not follow that any n-gon with $n \geq 5$ is not coverable. In particular, let AB be the longest side of the polygon. Considering a circle with diameter AB, and choose one of the two arcs defined with endpoints at A and B. Pick $n-2$ points $X_1, X_2, \ldots, X_{n-2}$ arbitrarily on that arc; the polygon defined by the resulting n points $A, X_1, X_2, \ldots, X_{n-2}, B$ is entirely covered by the circle with diameter AB and is therefore coverable.

6. Prove that $\tan 1°$ is an irrational number.

SOLUTION. First, consider angles α and β such that $\tan \alpha$ and $\tan \beta$ are both rational. Then it follows that

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

must also be a rational number.

Now let us assume by contradiction that $\tan 1°$ is a rational number. Applying the above observation on $\beta = 1°$ yields that $\tan(\alpha + 1°)$ is rational if $\tan \alpha$ is also rational, from which inducting down implies that $\tan x°$ is rational for all integers x. But plugging in $x = 60$ results in $\sqrt{3} = \tan 60°$ being rational, a contradiction.

7. Let a_1, a_2, a_3, b_1, b_2, b_3 be arbitrary positive numbers. Prove that

$$(a_1^7 + a_2^7 + a_3^7)(b_1^7 + b_2^7 + b_3^7) \geq (a_1^4 b_1^3 + a_2^4 b_2^3 + a_3^4 b_3^3)(a_1^3 b_1^4 + a_2^3 b_2^4 + a_3^3 b_3^4).$$

SOLUTION. Define $x_i = a_i^7$ and $y_i = b_i^7$ for any $i \in \{1, 2, 3\}$. Applying Hölder's Inequality on the x_i and y_i with $\frac{3}{7}$ and $\frac{4}{7}$ being the respective exponents, we find that

$$(x_1 + x_2 + x_3)^{\frac{3}{7}}(y_1 + y_2 + y_3)^{\frac{4}{7}} \geq \sqrt[7]{x_1^3 y_1^4} + \sqrt[7]{x_2^3 y_2^4} + \sqrt[7]{x_3^3 y_3^4}.$$

Similarly, applying Hölder's Inequality on the x_i and y_i again, but this time using $\frac{4}{7}$ and $\frac{3}{7}$ as the respective exponents, it then follows that

$$(x_1 + x_2 + x_3)^{\frac{4}{7}}(y_1 + y_2 + y_3)^{\frac{3}{7}} \geq \sqrt[7]{x_1^4 y_1^3} + \sqrt[7]{x_2^4 y_2^3} + \sqrt[7]{x_3^4 y_3^3}.$$

Multiplying the two above inequalities then gives

$$(x_1+x_2+x_3)(y_1+y_2+y_3) \geq \left(\sqrt[7]{x_1^3 y_1^4} + \sqrt[7]{x_2^3 y_2^4} + \sqrt[7]{x_3^3 y_3^4}\right)\left(\sqrt[7]{x_1^4 y_1^3} + \sqrt[7]{x_2^4 y_2^3} + \sqrt[7]{x_3^4 y_3^3}\right).$$

Substituting in $x_i = a_i^7$ and $y_i = b_i^7$ into the above then gives

$$(a_1^7 + a_2^7 + a_3^7)(b_1^7 + b_2^7 + b_3^7) \geq (a_1^4 b_1^3 + a_2^4 b_2^3 + a_3^4 b_3^3)(a_1^3 b_1^4 + a_2^3 b_2^4 + a_3^3 b_3^4),$$

as desired.