Berkeley Math Circle: Monthly Contest 4 Solutions

1. Let S be the region in the Cartesian plane containing exactly the points satisfying

$$\begin{aligned} y &\geq 5x - 20, \\ y &\geq 0, \\ y &\geq 2 - x, \\ y &\leq 2x + 2, \\ 3y - 20 &\leq -x. \end{aligned}$$

   Let $f(x, y) = 2x - 3y$. Find the minimum and maximum of f as it ranges across S.

   SOLUTION. Observe that the system of five inequalities above defines a convex pentagon P in the Cartesian plane. We claim that f reaches its minimum and maximum at vertices of P.

   By way of contradiction, assume first that the minimum is reached at some point (x_0, y_0) strictly in the interior of P. Since $f(x, y)$ is a linear function, it is linear along any straight line passing through (x_0, y_0). Now consider any line crossing the boundary of P twice, say at (x_1, y_1) and (x_2, y_2). Then by linearity at least one of $f(x_1, y_1)$ and $f(x_2, y_2)$ will be at least the value of $f(x_0, y_0)$. If both values are equal to $f(x_0, y_0)$, it follows that f is constant everywhere along the line, so that the minimum is still attained at (x_0, y_0); otherwise, exactly one of $f(x_1, y_1)$ and $f(x_2, y_2)$ will be smaller than $f(x_0, y_0)$, a contradiction.

   Observe that a similar argument holds if the same minimum is attained on the boundary of P but not at the vertices, and analogously in the case of maxima. In particular, f attains its extrema at the vertices of P.

   One can compute by graphing and intersecting the equations of the lines of the adjacent sides that the five vertices of P are $(5, 5)$, $(4, 0)$, $(2, 0)$, $(0, 2)$, and $(2, 6)$, with $f(5, 5) = -5$, $f(4, 0) = 8$, $f(2, 0) = 4$, $f(0, 2) = -6$, and $f(2, 6) = -14$. From here, it thus follows that the minimum of f on P is -14, attained at $(2, 6)$, while the maximum is 8 and is attained at $(4, 0)$.

2. Let a be a three-digit number with middle digit 0. Another three-digit number b is formed by reversing the digits of a. Prove that $a + b$ cannot be a perfect square.

   SOLUTION. By construction, there must exist digits m and n with $a = 100m + n$ and $b = m + 100n$, so that $a + b = 101(m + n)$. Since 101 is prime and $m + n \leq 18$, it follows that $101 \mid a + b$ but $101^2 \nmid a + b$, so $a + b$ cannot be a perfect square.

3. Let $ABCD$ be a cyclic quadrilateral with ω and O being its circumcircle and circumcenter, respectively. Suppose that $AC \perp BD$. Let P be the intersection of the diagonals AC and BD Given that $AP = DP = 1$ and $BP = CP = 2$, compute the area bounded by BP, CP, and ω in terms of $\angle BOC$, represented in radians.

SOLUTION. We first scale up our quadrilateral by a factor of 2 and place our points on the Cartesian plane as $P' = (0,0)$, $A' = (-2,0)$, $B' = (0,4)$, $C' = (4,0)$, and $D' = (0,-2)$. One can check by inspection that the circumcenter O' of $A'B'C'D'$ is at $(1,1)$ and that ω has radius $\sqrt{10}$.

The area of the desired scaled region is the area of sector $B'O'C'$ plus the area of the two triangles $\triangle P'O'B'$ and $\triangle P'O'C'$, which evaluates to

$$\begin{aligned}[B'O'C'] + [\triangle P'O'B'] + [\triangle P'O'C'] &= \left(\frac{\angle B'O'C'}{2\pi}\right)\left(\pi(\sqrt{10})^2\right) + \frac{4\cdot 1}{2} + \frac{1\cdot 4}{2} \\ &= 5\angle BOC + 4,\end{aligned}$$

where $\angle B'O'C' = \angle BOC$ since scaling preserves angles. Rescaling then implies that our original desired area is $\boxed{\frac{5\angle BOC}{4} + 1}$.

4. What is the period of the $\frac{2023_6}{114202_6}$ when written as a repeating decimal, when expressed in base 6? Here, the subscript of 6 represents base-6 notation.

SOLUTION. For notational clarity, all numbers will be in base 10 unless otherwise noted by a subscript.

We begin by converting the fraction to base 10, giving

$$\frac{2023_6}{114202_6} = \frac{447}{10010} = \frac{3\cdot 149}{2\cdot 5\cdot 7\cdot 11\cdot 13}.$$

Now observe that any repeating decimal will have a finite number of digits followed by an infinitely repeating segment of minimal length k so that

$$\frac{447}{10010} = \frac{m_0}{6^m} + \sum_{i=1}^{\infty}\frac{k_0}{6^{m+ik}} = \frac{m_0}{6^m} + \frac{k_0}{6^m(6^k-1)} = \frac{k_0 + (6^k-1)m_0}{6^m(6^k-1)},$$

where k is minimal and positive, m, m_0, and k_0 are nonnegative integers with $k_0 > 6^k$. Taking modulo $6^k - 1$, any positive integer m_1 can be written in the form $k_0' + (6^{k'}-1)m_0'$ for some fixed k', nonnegative $k_0' < 6^{k'}$, and nonnegative m_0'. Hence by minimality it suffices to find the minimal k for which there exists some m_1 with

$$\frac{m_1}{6^m(6^k-1)} = \frac{447}{10010}.$$

Since $\gcd(447, 10010) = 1$ and $2 \mid 6^m$, this occurs iff $5005 \mid 6^k - 1$, so it suffices to find the minimal k satisfying $5005 \mid 6^k - 1$.

Since $5005 = 5\cdot 7\cdot 11\cdot 13$, it follows that

$$k = \operatorname{lcm}(\operatorname{ord}_5(6), \operatorname{ord}_7(6), \operatorname{ord}_{11}(6), \operatorname{ord}_{13}(6)),$$

where $b = \operatorname{ord}_p(a)$ is the minimal integer exponent such that $a^b \equiv 1 \pmod{p}$. Since $6 \equiv 1 \pmod 5$ and $6^2 \equiv 1 \pmod 7$, we get $\operatorname{ord}_5(6) = 1$ and $\operatorname{ord}_7(6) = 2$. For the case of $p = 11$, Fermat's Little Theorem yields that $6^{10} \equiv 1 \pmod{11}$ and thus $\operatorname{ord}_{11}(6) \mid 10$. We can check that 6^1, 6^2, and 6^5 are all not 1 modulo 10, so $\operatorname{ord}_{11}(6) = 10$. Similarly, if $p = 13$, we can check that $\operatorname{ord}_{13}(6) = 12$. Thus

$$k = \operatorname{lcm}(10, 12, 2) = \boxed{60}.$$

5. Aerith and Bob play a game on an infinite lattice. On each of their turns, they may write 14 or 16 on a square of the lattice, 8 or 10 on a vertex, or an integer between 1 and 5, inclusive, on an edge. Bob wins if there are four numbered edges each adjacent to a numbered vertex whose sum is less than or equal to the number on the vertex, and where no more than two edges share a number. Aerith wins if there are four numbered edges each adjacent to a numbered square whose sum is greater than or equal to the number on the square, and where no more than two edges share a number. Show that Aerith can always stop Bob from winning if she goes first.

SOLUTION. Suppose by contradiction that assume that Bob has a winning strategy when Aerith goes first.

First, claim that if Bob has a winning strategy when Aerith goes first, then Bob has a winning strategy when he goes first. In particular, Bob could start by placing the number 10 on a vertex and make moves by ignoring the 10 as if he had gone second. Since a 10 on a vertex is strictly better than an 8 for Bob and Aerith's win condition does not care about number on vertices, it follows that the 10 will never sabotage Bob's plane; indeed, even if his winning strategy involved playing on that vertex, but then he could just write a 10 on a different vertex and pretend to ignore the new 10, since the grid is infinite and a 10 is more optimal than an 8 for Bob.

It thus suffices to prove that if Bob has a winning strategy going first, then Aerith also has a winning strategy going first, as this would contradict our original assumption that Bob has a winning strategy when Aerith goes first.

To this end, consider the infinite dual lattice where its vertices correspond to squares of the game lattice, edges correspond to edges, and squares correspond to vertices. Set up a one-to-one correspondence between games on the dual lattice and the original lattice by stating that the placement of a number k on an edge of the dual lattice corresponds to the placement of $6-k$ on the corresponding edge of the original, and the placement of a number n on a vertex or square corresponds to the placement of $24-n$ on the corresponding square or vertex of the original lattice, respectively.

Furthermore, state that Bob, instead of Aerith, wins on the dual lattice if there are four numbered edges, no more than two of which share a number, each adjacent to a numbered square on the dual lattice whose sum is greater than or equal to the number on the square, and that Aerith, instead of Bob, wins if there are four numbered edges, no more than two of which share a number, each adjacent to a numbered vertex whose sum is less than or equal to the number on the vertex.

In particular, observe that a game on the dual lattice and a game on the original lattice correspond exactly to one another. Therefore, if Bob has a winning strategy going first on the original lattice, he has a winning strategy going first on the dual lattice by playing the corresponding moves to his strategy on the original lattice, which in turn implies that Aerith has a winning strategy going first on the original lattice since she wins on the original lattice iff Bob wins on the dual lattice. This is our desired contradiction.

6. Let p be an odd prime, and define the polynomial $f(x) = x^{p+1} + (1-p)x^p - p$.

   (a) Prove that $x+1$ divides $f(x)$.

   (b) Let $g(x) = \frac{f(x)}{x+1}$. Prove that $g(x)$ is irreducible.

SOLUTION.

(a) Since p is odd, note that

$$f(-1) = (-1)^{p+1} + (1-p)(-1)^p - p = 1 + (1-p)(-1) - p = 1 + (p-1) - p = 0.$$

Hence -1 is a root of f, so it follows that $x - (-1) = x + 1$ must divide f.

(b) Note that

$$f(x) = x^{p+1} + (1-p)x^p - p = (x^{p+1} + x^p) - (px^p + p) = x^p(x+1) - p(x^p + 1).$$

Since p is odd, we use the factorization $x^p + 1 = (x+1)(x^{p-1} - x^{p-2} + \cdots - x + 1)$ to get that

$$g(x) = \frac{f(x)}{x+1} = \frac{x^p(x+1) - p(x^p+1)}{x+1} = x^p - p(x^{p-1} - x^{p-2} + \cdots - x + 1).$$

Letting a_i be the coefficient of the unique monomial degree of degree i in the $g(x)$ expression above, we observe that $|a_i| = p$ for all $i \in \{0, 1, \cdots, p-1\}$ and $a_p = 1$. In particular, we note that $p^2 \nmid a_0$ and $p \nmid a_p$, implying by Eisenstein's Criterion on the prime p that $g(x)$ is irreducible, as desired.

7. A *toroidal helix* is a curve $c(t)$ on a torus so that each angle function is a linear function of t. Note that a torus, thought of as a circle revolved around a line, has two angle functions, namely the angle of revolution and the angle on the circle.

(a) How many ways can one place a rectangular grid with 130 squares on a torus? The rectangular grid is generated by dividing the torus into 130 regions, defined by two families of non-intersecting toroidal helices that always meet at 90°.

(b) What about a hexagonal grid with 105 hexagons on a torus? The hexagonal grid is generated similarly, using 3 families of non-intersecting toroidal helices that always meet at 60°.

Two grids are considered the same if the induced graph structures are equivalent, so that there is a one-to-one correspondence between regions that preserves adjacency.

Hint: read about universal covering manifolds, flat tori, and Dedekind's second proof (1894) of Fermat's Theorem on sums of squares.

SOLUTION.

(a) The problem asks to find the number of ways to place a square grid with 130 squares on a torus. For clarity, we present a step-by-step approach.

Step 1. First, unfold the torus into a rectangle as depicted `here`.

Step 2. Now, take this rectangle and place it in the coordinate plane, according to the grid we created.

Step 3. From here, we can see that the torus is determined by the position of the four corners of this rectangle in the plane. In fact, the grid on the torus is uniquely determined by the position of these four corners up to Euclidean motions (translation, rotation, reflection).

Step 4. Because we don't want to overcount, we note that the grid on the torus is uniquely determined by this rectangle, up to Euclidean motion (i.e. translation, rotation, reflection). For this, we move some vertex to the origin and write the adjacent vertices as (a, b) and (c, d). From here, the requirement that they are orthogonal can be written as $ac + bd = 0$ and the area is given by $ad - bc = 130$.

Step 5. The key observation is that this can be written in complex numbers as

$$(a + bi)(d + ci) = 130$$

Thus, we only need to find the factors of 130 over the complex integers.

Step 6. We claim that the prime factorization is

$$130 = (1 + i)(1 - i)(2 + i)(2 - i)(3 + 2i)(3 - 2i)$$

To see why, let $N(z) = z\overline{z}$ be the norm-squared of a complex number. If z is a complex integer then $N(z)$ is also a complex integer, and N also satisfies $N(wz) = N(w)N(z)$. Now, we can check by inspection that each of the factors listed has prime N, and that any further factorization would require a factor z with $N(z) = 1$, which can only occur when $z = \pm 1$ or $z = \pm i$. To see where we got these factors, just try to write each prime as a sum of squares. For example, $13 = 3^2 + 2^2 = (3 + 2i)(3 - 2i)$.

Step 7. We digress briefly to discuss prime factorization. Prime factorization is unique up to ordering of the primes and multiplication by ± 1. Over the complex integers, it works out much the same way, except it works up to multiplication by ± 1 and $\pm i$. Thus, 130 has 5 distinct prime factors, where $(1 + i) = i(1 - i)$ is repeated. Thus, while the total number of factors is $4 \cdot 3 \cdot 2^4 = 192$

Step 8. In particular, we would like to find the number of pairs (x, y) with $xy = 130$ where we consider pairs equivalent if they are conjugate, reversed, or change sign. That is,

$$(x, y),\ (ix, iy),\ (\overline{x}, \overline{y}),\ (y, x)$$

all generate the same grid on the torus. Let us say that a complex number is special if it is either real, pure imaginary, or its real and imaginary parts are equal up to sign. This occurs if it lies on the coordinate axes or it does when it is rotated 45°. We divide the factors into several cases:

1. x and y are not special and $x \neq \overline{y}$
2. Both x and y are special, and $x \neq \overline{y}$
3. The property $x = \overline{y}$ is satisfied.

Note that it is not possible for one of them to be special and the other is not, because the angle of x is the negative of the angle of y.

In the first case, there are 16 equivalent pairs because there are four choices of sign, and we could take the conjugate of both, and we could swap x and y, and all of these generate distinct rectangles.

In the second case, there are 8 equivalent pairs, because taking conjugates does not generate distinct pairs (they are equal up to ± 1 or $\pm i$). We can count the number of possibilities here by adding the 32 real/imaginary factors of 130 to the 16 factors that lie on the lines of slope $\pm 45^\circ$ to get a total of 48 factors, with 6 distinct pairs.

In the third case, we deduce that x is given by partitioning the prime factors of 130 into two conjugate subsets. There are are 4 ways to do this, and 16 ways allowing for multiplication by units (i.e. ± 1 and $\pm i$). Now, each of these generates 8 equivalent pairs unless x is special, which is not possible because that would require that 130 or $\frac{130}{2}$ is a perfect square, so we have 2 new distinct pairs. These come from the factorizations $130 = (11+3i)(11-3i)$ and $130 = (9+7i)(9-7i)$.

We have 192 factors in total, and $192-48-16 = 128$ pairs in the first case, of which $\frac{128}{16} = 8$ are distinct. Therefore, we have a total of $8+6+2 = \boxed{16}$ distinct rectangles.

(b) Now, for the hexagonal case, we give just a sketch. We work in coordinate given on a hexagonal grid and use the Eisenstein integers $\mathbb{Z}[\omega]$ instead of the complex (Gaussian) integers. The orthogonality condition is then given, in combination with the area condition, as $(a+b\omega)(c-d\omega) = \sqrt{32} \cdot 105i = 105(\omega + \frac{1}{2})$, which is not an Eisenstein integer. It follows that there are $\boxed{0}$ solutions because 105 is odd. On the other hand, if we had picked an even integer, we would proceed analogously to as in (a).