Berkeley Math Circle: Monthly Contest 3 Solutions

1. Solve, in positive real numbers, the equation $(x^y)^{2023} = x^{(y^{2023})}$.

   SOLUTION. Rewrite the equation as $x^{2023y} = x^{y^{2023}}$. Therefore, if $x = 1$, then any real positive y trivially satisfies it. Otherwise, if $x \neq 1$, then we take logarithms to find that y must satisfy $y \cdot 2023 = y^{2023}$. Dividing both sides by y yields $y^{2022} = 2023$, which means that $y = \sqrt[2022]{2023} = \boxed{2023^{\frac{1}{2022}}}$.

2. Find the number of complex integer solutions to the equation $x^2 - y^2 = 17$.

   SOLUTION. First we note that $x^2 - y^2 = (x + y)(x - y)$. Therefore, by setting $m = x + y$ and $n = x - y$, each solution to our equation is equivalent to an ordered pair (m, n) satisfying $mn = 17$. Observe that the pairs (m, n) and (x, y) uniquely biject to each other, as we can solve for x and y, finding that $x = \frac{m+n}{2}$ and $y = \frac{m-n}{2}$.

   Now, 17 factors completely as $(4+i)(4-i)$ over the complex integers. This is because the numbers $4 + i$ and $4 - i$ are both prime over the complex integers, which follows from the fact that the norm function $|a + bi| = \sqrt{a^2 + b^2}$ is multiplicative and that $|4+i| = |4-i| = 17$, which is prime. One can check that the only other factorization of 17 over the complex integers is the trivial product $1 \cdot 17$.

   Therefore, up to reordering, all the complex integer factor pairs of 17 are $(1, 7)$, $(4+i, 4-i)$, and their unit multiples, which are formed by multiplying one factor by 1, i, -1, or $-i$ and the other by 1, $-i$, -1, or i, respectively. Each of these 8 distinct factor pairs satisfies the condition that their sum and difference are each divisible by 2, with each unordered pair corresponding to two ordered pairs because 17 is not square.

   We thus have exactly 16 ordered pairs (m, n) such that $mn = 17$, with each ordered pair corresponding to exactly one solution to our equation by setting $x = \frac{m+n}{2}$ and $y = \frac{m-n}{2}$. Thus, our answer is $\boxed{16}$.

3. Kelly draws an arrow on a regular $(4n + 2)$-gon. She then reflects the $(4n + 2)$-gon about its $4n + 2$ lines of symmetry in some order. Prove that the arrow now points in a different direction.

   SOLUTION. Draw two $(2n + 1)$-gons, each using every other vertex. The $2n + 1$ lines of symmetry that pass through vertices take each $(2n + 1)$-gon to itself, and the other $2n + 1$ lines of symmetry take each $(2n + 1)$-gon to the other $(2n + 1)$-gon. Applying an odd number of such transformations will therefore result in a non-identity transformation.

4. Let $[y]$ denote the function which rounds a real number y to the nearest integer, with half-integers being rounded up. Prove that for any positive integer x,

$$\sum_{i=1}^{\infty} \frac{x}{2^i} = \sum_{i=1}^{\infty} \left[\frac{x}{2^i}\right].$$

SOLUTION. We have $\sum_{i=1}^{\infty} \frac{x}{2^i} = \frac{\frac{x}{2}}{1-\frac{1}{2}} = x$, so it suffices to prove that

$$\sum_{i=1}^{\infty} \left[\frac{x}{2^i}\right] = x.$$

Write x uniquely in the form $x = \sum_{j=0}^{n} e_j 2^j$ for some positive integer n, where each $e_j \in \{0,1\}$. For each i, it then follows that

$$\left[\frac{x}{2^i}\right] = \left[\sum_{j=0}^{n} \frac{e_j 2^j}{2^i}\right] = \left[\sum_{j=0}^{n} e_j 2^{j-i}\right] = \left[\sum_{j=i-1}^{n} e_j 2^{j-i} + \sum_{j=0}^{i-2} e_j 2^{j-i}\right].$$

Observing that the fractional portion of $\sum_{j=i-1}^{n} e_j 2^{j-i}$ is always either 0 or $\frac{1}{2}$ and also that $\sum_{j=0}^{i-2} e_j 2^{j-i} \le \sum_{j=0}^{i-2} 2^{j-i} = \sum_{j=0}^{i-2} 2^{-2-j} < \frac{1}{2}$, it follows that

$$\left[\frac{x}{2^i}\right] = \left[\sum_{j=i-1}^{n} e_j 2^{j-i} + \sum_{j=0}^{i-2} e_j 2^{j-i}\right] = \left[\sum_{j=i-1}^{n} e_j 2^{j-i}\right].$$

Every term in the summation $[\sum_{j=i-1}^{n} e_j 2^{j-i}]$ is an integer except the first term, so that

$$\left[\frac{x}{2^i}\right] = \left[\sum_{j=i-1}^{n} e_j 2^{j-i}\right] = \sum_{j=i-1}^{n} [e_j 2^{j-i}].$$

Hence

$$\sum_{i=1}^{\infty} \left[\frac{x}{2^i}\right] = \sum_{i=1}^{\infty} \sum_{j=i-1}^{n} [e_j 2^{j-i}] = \sum_{j=0}^{n} \sum_{i=1}^{j+1} [e_j 2^{j-i}]$$

after switching the order of summation.

For all j, we have $e_j \in \{0,1\}$, and thus $[e_j 2^{j-i}] = e_j [2^{j-i}]$ so we get

$$\sum_{i=1}^{\infty} \left[\frac{x}{2^i}\right] = \sum_{j=0}^{n} \sum_{i=1}^{j+1} [e_j 2^{j-i}] = \sum_{j=0}^{n} \sum_{i=1}^{j+1} e_j [2^{j-i}] = \sum_{j=0}^{n} e_j \left(\sum_{i=1}^{j+1} [2^{j-i}]\right).$$

However,

$$\sum_{i=1}^{j+1} [2^{j-i}] = \sum_{i=-1}^{j-1} [2^i] = 1 + \sum_{i=0}^{j-1} 2^i = 2^j,$$

yielding

$$\sum_{i=1}^{\infty} \left[\frac{x}{2^i}\right] = \sum_{j=0}^{n} e_j \left(\sum_{i=1}^{j+1} [2^{j-i}]\right) = \sum_{j=0}^{n} e_j 2^j = x,$$

as desired.

5. A deck of cards contains 52 cards labeled 1 through 52 in order from top to bottom. We may define a riffle shuffle as a process in which the deck is cut into two piles, and then the two piles are shuffled back together in a way such that the ordering of cards within each pile is preserved. Find the minimum number of consecutive riffle shuffles required to reverse the order of cards in the deck completely.

SOLUTION. The answer is 6.

First, we show that it must take at least 6 shuffles to reverse the order of the cards. Consider the longest ascending subsequence of the cards from top to bottom. After a riffle shuffle, the length of the longest ascending subsequence can be at minimum half the length of the longest ascending subsequence before the riffle shuffle, since at least half of the subsequence will be in one of the piles after the cut, and these piles remain in top-to-bottom order after the shuffle. However, the deck starts with a 52-card ascending subsequence, so it must take at least $\log_2(52) > \log_2(32) = 5$ riffle shuffles to get become fully reversed, as the longest ascending subsequence in the reversed deck has length 1.

Now, we show that the deck can be reversed in 6 shuffles. First, we note that any valid riffle shuffle on a deck is also a valid riffle shuffle on any subset of cards in that deck. Therefore, if we can completely reverse the order of cards in a 64-card deck in 6 shuffles, we may also do so in a 52-card deck simply by ignoring the last 12 cards of the 64-card deck.

We may do this by having each shuffle cut the 64-card deck exactly in half, and then interlacing the cards such that the top card of the bottom half ends up on top of the shuffled deck, the second card of the bottom half ends up third in the shuffled deck, and so on. Equivalently, each shuffle takes the card in the nth position of the deck to the $(2n)$th position of the shuffled deck, with indices taken modulo 65. This implies that the card labeled n will end up in the $(64n)$th position of the deck after 6 shuffles; however, since $2^6 = 64 \equiv -1 \pmod{65}$, this is the same as the $-n$th position, modulo 65. Thus, each card ends up being n cards from the bottom in the final deck, where n is its label, corresponding to a completely reversed deck. Thus, the deck can be completely reversed in $\boxed{6}$ riffle shuffles, as desired.

6. Let P be some n-gon inscribed in the unit circle. Prove that there exists vertices A, B, and C of P such that the area of triangle $\triangle ABC$ is at most $\frac{20}{n^2}$.

SOLUTION. Label the vertices of P as $X_1, X_2, \cdots, X_n$, where the X_i go in clockwise order. Let a_i be the sum of the lengths of the two sides having X_i as a vertex. Then $a_1 + a_2 + \cdots + a_n$ is exactly double the perimeter of P, so that

$$a_1 + a_2 + \cdots + a_n = 2p(P) \leq 2(2\pi) = 4\pi$$

as P is inscribed in a unit circle of circumference 2π. Thus, the Pigeonhole Principle implies that there must exist some a_k such that

$$a_k \leq \frac{4\pi}{n}.$$

Let AX_k and X_kC be the two segments surrounding X_k, so that $AX_k + X_kC \leq AC = a_k \leq \frac{4\pi}{n}$. Then the AM-GM Inequality gives

$$AX_k \cdot X_kC \leq \left(\frac{AX_k + X_kC}{2}\right)^2 \leq \left(\frac{2\pi}{n}\right)^2.$$

Using the sine triangle area formula, we find that, since $\sin\angle AX_kC < 1$,

$$[AX_kC] = \frac{AX_k \cdot X_kC \cdot \sin\angle AX_kC}{2} \leq \frac{AX_k \cdot X_kC}{2} \leq \frac{1}{2}\left(\frac{2\pi}{n}\right)^2 = \frac{2\pi^2}{n^2} \leq \frac{20}{n^2},$$

so setting $B = X_k$ implies the result.

7. In triangle $\triangle ABC$, let D, E, and F be points on sides BC, CA, and AB, respectively, such that $AD \perp BC$ and the cevians AD, BE, and CF concur. Prove that $\angle ADF = \angle ADE$.

   SOLUTION. Let X be the intersection of the lines AB and DE. By the projective Ceva-Menelaus Theorem, it follows that $(XF; BA) = -1$ is a harmonic bundle. Since $\angle BDA = \angle CDA = 90°$, the projective Angle Bisector Theorem then implies that AD bisects $\angle FDE$, so that $\angle ADF = \angle ADE$, as desired.