Berkeley Math Circle: Monthly Contest 2 Solutions

1. Bob has had a bad experience with the laundry machine and now has three black socks, three blue socks, and three red socks in his sock drawer. He draws three socks from the drawer, one at a time, without placing any sock that had been drawn back in the drawer. What is the probability that at least two of these three socks share the same color?

   SOLUTION. The only way that for Bob to not get two socks of the same color is for all three socks to be of different colors. No matter what sock he draws first, there is a probability of $\frac{9-3}{9-1} = \frac{3}{4}$ that the second sock does not share a color with the first, and, given that the first two socks are of different colors, a probability of $\frac{9-3-3}{9-2} = \frac{3}{7}$ that the third does not share a color with either of the first two. Therefore, there is a $\frac{3}{4} \cdot \frac{3}{7} = \frac{9}{28}$ probability that Bob draws three different colors of socks, so the probability that he draws at least two socks with the same color is $1 - \frac{9}{28} = \boxed{\frac{19}{28}}$.

2. Certain positive integers n have the property that, for all positive even numbers m, the last two digits of mn and the last two digits of m are exactly the same. In fact, there are exactly two positive integers less than 100 with this property. What are they?

   Note that numbers below 10 are treated as having two digits. For example, the last two digits of 8 are said to be 08.

   SOLUTION. Let n satisfy the given property. Then $2kn \equiv 2k \pmod{100}$, where $2k$ is any positive two-digit even number, so that $2k(n-1) \equiv (2k-2)(n-1) \equiv 0 \pmod{100}$. Thus $2(n-1) \equiv 0 \pmod{100}$, which yields that $n \equiv 1 \pmod{50}$. The only possible values for positive n under 100 are thus $\boxed{1 \text{ and } 51}$, and it is easy to see that they both exhibit the desired property.

3. Define points $A = (0,0)$, $B = (0,5)$, $C = (3,0)$, and $D = (3,5)$ on the coordinate plane. How many circles of radius 1 can be drawn entirely within rectangle $ABDC$ without overlapping?

   SOLUTION.

   The answer is $\boxed{2}$. A corresponding construction can be achieved with unit circles centered at $(1,1)$ and $(1,3)$.

   Suppose by contradiction that one can fit 3 nonoverlapping unit circles in rectangle $ABDC$. Let O_1, O_2, and O_3 be their centers. Then each of the O_i must be at least 1 unit away away from the edge of the rectangle, so they must all lie in the rectangle $A'B'D'C'$ where $A' = (1,1)$, $B' = (1,4)$, $C' = (2,1)$, and $D' = (2,4)$.

   Let M and N respective be the midpoints of $A'B'$ and $C'D'$. Then at least two of the O_i must lie in the interior or on the boundary of either rectangle $A'MNC'$ or rectangle $B'D'NM$. Suppose without loss of generality that O_1 and O_2 lie in or on rectangle $A'MNC'$. But this implies that

$$O_1O_2 \le A'N = \sqrt{A'M^2 + MN^2} = \sqrt{\left(\frac{3}{2}\right)^2 + 1^2} = \frac{\sqrt{13}}{2} < 2,$$

which means that the unit circles centered at O_1 and O_2 overlap, a contradiction.

4. A straight line segment of length 1 is given in the plane. Draw a line segment of length $\sqrt{\sqrt{5} - 2}$ using only a compass and a straightedge.

SOLUTION. Let us say that a positive real number a is *constructible* if it is possible to construct a segment of length a under the given setup. Note in particular that if a and b are constructible numbers, then $|a - b|$ is also constructible. Indeed, one can

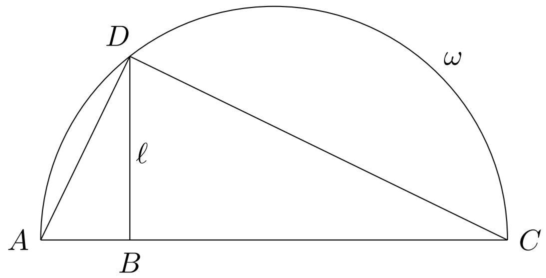
- construct line segment XY of length a, and
- construct line segment XZ of length b such that Y and Z lie on the same side of X.

Then segment YZ must have length $|a - b|$.

We also demonstrate that if a is constructible, then $\sqrt{a}$ is too. Assuming familiarity with basic constructions, such as that of a line perpendicular to a segment at a given point or the midpoint of any given segment, given a segment of length a, one can

- construct line AC of length $a + 1$, with point B on AC such that $AB = a$,
- construct the circle ω of diameter AC by constructing the midpoint M of AC,
- construct the line ℓ perpendicular to AC at B,
- construct point D given by one of the intersections of ω and ℓ.

Then $\triangle ABD \sim \triangle ADC \sim \triangle DBC$, so that $BD = \sqrt{AB \cdot BC} = \sqrt{a}$.



Using the above results, we have that 5 and 2 are constructible, so $\sqrt{5}$ is constructible and therefore $\sqrt{5} - 2$ is too. It then follows that $\sqrt{\sqrt{5} - 2}$ must be constructible too, as desired. In fact, applying the above procedures in the appropriate order yields a concrete method for constructing this length.

As a remark, it turns out that if a and b are constructible, then so are $a + b$, $a - b$, ab, $\frac{a}{b}$, and $\sqrt{a}$. A result in abstract algebra in fact states that the set of constructible integers are closed under addition, subtraction, multiplication, division, and square roots, with no other operations being possible. For instance, lengths such as $\sqrt{\sqrt{5} - 2}$ and $\sqrt{\frac{5}{41} + 3\sqrt{17 + \sqrt[4]{999}}}$ as constructible, but $\sqrt[3]{2}$ and π are not.

5. Find all nonzero polynomials $P(x)$ satisfying $x^3P(P(x^3)) = P(x^4P(x^2))$ for all real numbers x.

SOLUTION. Setting $n = \deg P$, the given condition implies that $3n^2+3 = 2n^2+4n$, which reduces to having either $n = 1$ or $n = 3$.

If $n = 1$, then there exist constants $a, b \in \mathbb{R}$, with $a \neq 0$, for which $P(x) = ax + b$. Plugging in $x = 0$ yields $b = 0$, so that $P(x) = ax$. One can verify that all such polynomials of this form work, regardless of the nonzero value of a.

If $n = 3$, then we have constants $a, b, c, d \in \mathbb{R}$, with $a \neq 0$, such that $P(x) = ax^3 + bx^2 + cx + d$. Again, putting $x = 0$ into the given relation implies that $d = 0$, so that $P(x) = ax^3 + bx^2 + cx$.

Define $Q(x) = x^3P(P(x^3)) = P(x^4P(x^2))$, and observe that if the coefficient of x^m is nonzero in Q, then we must have $3 \mid m$. In particular, since

$$\begin{aligned} Q(X) &= P(x^4P(x^2)) = P(ax^{10} + bx^8 + cx^6) \\ &= a(ax^{10} + bx^8 + cx^6)^3 + b(ax^{10} + bx^8 + cx^6)^2 + c(ax^{10} + bx^8 + cx^6), \end{aligned}$$

the coefficient of the x^{10} term in Q is ac, implying that $c = 0$ and therefore that

$$Q(x) = a(ax^{10} + bx^8)^3 + b(ax^{10} + bx^8)^2.$$

In turn, the coefficient of the x^{16} term in Q is b^3, implying that $b = 0$. Hence $P(x) = ax^3$, and one can also verify that all nonzero values of a produce a valid polynomial P.

The solution set for P is therefore $\boxed{\{ax, ax^3 : a \in \mathbb{R} \setminus \{0\}\}}$.

6. A *multiset* is an unordered set where an element of the set can appear multiple times. We call a multiset of integers, all strictly greater than 1, *a division partition of x* if the elements of the multiset multiply to x. For any positive integer n, let $f(n)$ be the number of distinct division partitions of n. For instance, the multiset $\{2, 2, 2, 3, 30\}$ is a division partition of 720, and one can check that $f(100) = 9$. Show that $f(10^{10}) + f(40^5) + f(250^5) < 2.5^{19}$.

SOLUTION.

For nonnegative integers x and y, let $p(x, y)$ be the number of division partitions of 2^x5^y, and let $p'(x, y)$ be the number of division partitions of 2^x5^y that are disjoint from the set $\{2, 5\}$. If either $\min\{x, y\} < 0$, we write $p(x, y) = p'(x, y) = 0$.

One may verify that, as long as $2^{x-a}5^{y-a}$ is an integer greater than 1, the number of division partitions of 2^x5^y that contain a copies of 2 and b copies of 5 is simply $p(x - a, y - b)$ by bijecting the two corresponding sets of division partitions. Hence, making the appropriate inclusions and exclusions, as long as $\max\{x, y\} \geq 1$, the number of division partitions of 2^x5^y containing at least one of 2 or 5 is then $p(x - 1, y) + p(x, y - 1) - p(x - 1, y - 1)$. This value is also equal to $p(x, y) - p'(x, y)$, so we find that, for all nonnegative x and y with $\max\{x, y\} \geq 1$, we have

$$p(x, y) = p(x - 1, y) + p(x, y - 1) - p(x - 1, y - 1) + p'(x, y).$$

Now, assume that $x+y \geq 5$. Note that each division partition of $2^x 5^y$ that does not include the number 2 or 5 is either exactly $\{2^x 5^y\}$ or contains a minimal element $2^{x-j}5^{y-k}$ with $2 \leq (x-j)+(y-k) \leq \frac{x+y}{2} < x+y-2$, so that $3 \leq j+k \leq x+y-2$. Let $S(x,y)$ be the set containing all integer pairs (j,k) such that $0 \leq j \leq x$, $0 \leq k \leq y$, and $3 \leq j+k \leq x+y-2$. Since $p'(j,k)$ is an upper bound for the number of division partitions with lowest element $2^{x-j}5^{x-k}$, it follows that

$$\begin{aligned}
p'(x,y) - 1 &\leq \sum_{(j,k)\in S(x,y)} p'(j,k) \\
&= \sum_{(j,k)\in S(x,y)} (p(j,k) + p(j-1,k-1) - p(j,k-1) - p(j-1,k)) \\
&= p(x,y-2) + p(x-1,y-1) + p(x-2,y) \\
&\qquad - p(x-1,y-2) - p(x-2,y-1) \\
&\qquad + p(0,1) + p(1,0) - p(2,0) - p(1,1) - p(0,2).
\end{aligned}$$

where the middle line follows from our previous result on $p'(x,y)$, as $j+k \geq 3$ and thus $\max\{j,k\} \geq 1$ for all $(j,k) \in S(x,y)$, and the final line follows from telescoping sums, since any other terms are canceled out. Manual computation yields $p(0,1)+p(1,0)-p(2,0)-p(1,1)-p(0,2) = 1+1-2-2-2 = -4$. Thus

$$p'(x,y) \leq p(x,y-2)+p(x-1,y-1)+p(x-2,y)-p(x-1,y-2)-p(x-2,y-1)-3.$$

Since $x+y \geq 5$, it follows that

$$\begin{aligned}
p(x,y) &= p(x-1,y) + p(x,y-1) - p(x-1,y-1) + p'(x,y) \\
&\leq p(x-1,y) + p(x,y-1) + p(x,y-2) + p(x-2,y) \\
&\qquad - p(x-1,y-2) - p(x-2,y-1) - 3.
\end{aligned}$$

Define the function $s(n) = \sum_{i=0}^{i=n} p(i,n-i)$ for all positive integers n. We prove by induction that the function $r(n) = 2.5^n - s(n)$ is positive and monotonically increasing.

For the base case, we manually compute $s(2) = 6, s(3) = 14$, and $s(4) = 33$. One can check that this works, and that $r(2) = 2.5^2 - s(2) > 0$.

For the inductive step, let $n \geq 5$. Then

$$\begin{aligned}
s(n) &= \sum_{i=0}^{i=n} p(i,n-i) \\
&< \sum_{i=0}^{i=n} (p(i-1,n-i) + p(i,n-i-1) + p(i,n-i-2) + p(i-2,n-i) \\
&\qquad - p(i-1,n-i-2) - p(i-2,n-i-1)) \\
&= 2s(n-1) + 2s(n-2) - 2s(n-3),
\end{aligned}$$

where the second line follows from the recursive bound on $p(x,y)$ as derived above, dropping the -3 term.Then

$$\begin{aligned}
r(n) &> 2.5^n + 2(-2.5^{n-1} + r(n-1) - 2.5^{n-2} + r(n-2) + 2.5^{n-3} - r(n-3)) \\
&= 2.5^{n-3}(2.5^3 - 2\cdot 2.5^2 - 2\cdot 2.5 + 2) + r(n-1) + (r(n-2) - r(n-3)) \\
&\geq 2.5^{n-3}(0.125) + r(n-1),
\end{aligned}$$

where the last line follows from the inductive hypothesis. Thus $r(n) > r(n-1)$ as desired.

We use the above to show that $p(5,15) + p(10,10) + p(15,5) < 2.5^{19}$. In particular, our previous bound on $p(x,y)$ gives

$$\begin{aligned} p(5,15) &< p(4,15) + p(5,14) + p(5,13) + p(3,15) - p(4,13) - p(3,14) \\ &< p(3,16) + p(4,15) + p(5,14) + p(6,13) \end{aligned}$$

and similarly we have

$$p(10,10) < p(8,11) + p(9,10) + p(10,9) + p(11,8)$$

and

$$p(15,5) < p(13,6) + p(14,5) + p(15,4) + p(16,3).$$

Summing the above bounds, we find that

$$p(5,15) + p(10,10) + p(15,5) < \sum_{i=3}^{16} p(i, 16-i) < s(19) < 2.5^{19}$$

as $r(19) > 0$, so we are done.

7. Prove that there exist two real numbers $a, b > 1$ such that there are infinitely many pairs of positive integers m, n such that $0 < a^m - b^n < 1$.

SOLUTION. Let $b = 2$. We make $a = 2^x$ for a value of x that we inductively construct that will take on the form

$$x = 1 + \sum_{i=1}^{N} 10^{-c_i},$$

where $c_1, c_2, c_3, \ldots$ is an increasing sequence of positive integers and N is arbitrarily large. We aim for $a^{10^k} = 2^{10^k x}$ to always be at most 1 more than a power of two whenever $10^k x$ has a 1 in its units place. This can be achieved by working out the decimal representation of x from left to right, each time adding sufficiently many zeros before the next one, as $10^k x$ can get arbitrarily close to an integer by doing so. Letting x be the limit of this process gives a working pair (a, b).