Berkeley Math Circle: Monthly Contest 1 Solutions

1. How many positive integer factors of 10^{10} are not factors of 8^{10}? For example, 1, 2, 3, 4, 6, and 12 are all positive integer factors of 12, but 7 and 9 are not. Note that both 1 and 10^{10} count as valid factors of 10^{10}.

   SOLUTION. The prime factorizations for 10^{10} and 8^{10} are respectively $2^{10}5^{10}$ and 2^{30}. Hence, the factors of 10^{10} are of the form 2^i5^j, where $0 \leq i \leq 10$ and $0 \leq j \leq 10$, while the factors of 8^{10} are of the form 2^k, where $0 \leq k \leq 30$. The factors of 10^{10} that are not factors of 8^{10} must be of the form 2^i5^j, where $0 \leq i \leq 10$ and $0 < j \leq 10$, yielding 11 possibilities for i and 10 possibilities for j. As a result, there are $11 \cdot 10 = 110$ options for (i, j), with each possible pair (i, j) giving exactly one factor of 10^{10} satisfying the given conditions since all positive integers can be uniquely factorized into primes. The answer is therefore $\boxed{110}$.

2. Four distinct points A, B, C, and D lie on a plane. Can it be that the distances AB, AC, AD, BC, BD, and CD, in some order, are

   (a) 1, 1, 1, 1, 1, 2?

   (b) 1, 2, 3, 4, 5, 6?

   SOLUTION.

   (a) The answer is $\boxed{\text{no}}$. Indeed, three of the four points must form an equilateral triangle of side length 1. Let these be A, B, and C, without loss of generality. The distances between D, and two other points, say A and B, must then also be 1. In particular, both $\triangle ABC$ and $\triangle ABD$ are equilateral triangles, with C and D on the opposite sides of line AB. But this implies that $CD = \sqrt{3} \notin \{1, 2\}$, a contradiction.

   (b) The answer is $\boxed{\text{yes}}$. For instance, impose a coordinate system on the plane, and set $A = (0, 0)$, $B = (1, 0)$, $C = (4, 0)$, and $D = (6, 0)$, so that $AB = 1$, $CD = 2$, $BC = 3$, $AC = 4$, $BD = 5$, and $AD = 6$, as desired.

3. A castle has infinitely many rooms labeled $1, 2, 3, \ldots$, which are divided into some number of halls. It is known that room n is on the same hall as rooms $3n + 1$ and $n + 81$ for every n. At most how many halls can this castle have?

   SOLUTION. Let us say that two rooms are *connected* if they are located on the same hall. For every n, we observe that

   - room n connects to room $3(3(3(3n + 1) + 1) + 1) + 1 = 81n + 40$,
   - room $81n + 40$ connects to room $(81n + 40) + 81 = 81(n + 1) + 40$, and that
   - room $81(n + 1) + 40 = 3(3(3(3(n + 1) + 1) + 1) + 1) + 1$ connects to room $n + 1$.

   In particular, it follows that rooms n and $n + 1$ are connected for all n, so the castle has exactly $\boxed{1}$ hall.

4. Which is bigger, $A = 1^{100} + 2^{100} + \cdots + 99^{100}$, or $B = 100^{100}$?

SOLUTION. Observe that for $0 < n \leq 99$, we have

$$\left(\frac{100-n}{100}\right)^{100} = \left(1 - \frac{n}{100}\right)^{100} < \left(e^{-\frac{n}{100}}\right)^{100} = e^{-n}.$$

Thus,

$$\frac{A}{B} < e^{-1} + e^{-2} + \cdots + e^{-99} < \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{99}} = 1 - \frac{1}{2^{99}} < 1,$$

so $\boxed{B}$ is bigger.

5. There are 100 bags of money, numbered from 1 to 100. The bag with number n contains \$$n$ for all n. Then, 100 people are asked one by one which bag they would like to get money from, and can pick based on the responses of everyone who came before. Once an answer has been collected from everyone, each bag will be split equally among the people who picked it. If they are all playing optimally so as to maximize their individual profit, how many bags will not be picked?

SOLUTION. If there were some bag m which a participant could switch to after the game finished to increase their profit, if they instead picked m on their turn, each future player would have strictly less incentive to pick m, and at least as much incentive to pick every bag other than m. Thus by the end of the process, no participant would be better off if they were granted the ability to switch answers.

We claim that the final state will be that all bags numbered 34 to 67 will be picked once, and bags 68 to 100 will be picked twice. Otherwise, some bag n would have been picked by more than $\lfloor \frac{n}{34} \rfloor$ people, while some other bag m would have been picked by less than $\lfloor \frac{n}{34} \rfloor$ people. But then someone who picked n should have picked m instead so as to earn at least \$34, as they ended up with less than that. Thus the answer is $\boxed{33}$.

6. Let $C(p)$ be an annual calendar for year p. Two calendars for years p and q are said to be *identical* if every date on both calendars falls on the same day of the week, and we denote this as $C(p) = C(q)$. For example, $C(2021)$ and $C(2027)$ are identical, because January 1 on both calendars falls on a Friday, and both years are non-leap years. Call the smallest positive integer N for which $C(p) = C(p+N)$ for all p the *cicada period* of the calendar system C.

   (a) Is the Julian calendar system C_J periodic? If so, what is its cicada period? In the Julian calendar, a year is a leap year if and only if it is divisible by four.

   (b) Repeat the above part for the Gregorian calendar system C_G. It features a more elaborate and accurate definition of leap years, in which p is a leap year if and only if either p is divisible by four but not by 100, or that p is divisible by 400. For example, the years 1700, 1800, and 1900 are not leap years, but the years 1600 and 2000 are.

SOLUTION.

(a) Observe that there are two reasons why the calendars of certain years are non-identical. First, note that the number of days in a year, which is either 365 or 366, is not divisible by seven. In particular, the January 1st of different years will fall on the seven different days of the week, giving seven different flavors of calendars. Hence, the cicada period of the calendar system C' that ignores leap years would be 7.

However, leap years complicate the picture, as even if the January 1st of two different years p and q fall on the same day of the week, the March 1st of those years might not. For example, observe that $C_J(2021) = C_J(2027)$, as January 1 on each calendars falls on a Friday, and both are non-leap years. However, we also have that $C_J(2022) \neq C_J(2028)$, as 2021 is the first year after the leap year 2020, while 2027 is the third year after the leap year 2024.

The Julian calendar C_J defines the leap year in a way that quadruples the period of the calendar sequence. In particular, any two years p and q satisfying $C_J(p) = C_J(q)$ start new periods if and only if they are in the same positions within their four-year cycles, so that $p \equiv q \pmod 4$, in addition to having $p \equiv q \pmod 7$. This implies that $C_J(p) = C_J(q)$ if and only if $p \equiv q \pmod{28}$. Hence, the cicada period of C_J is $\boxed{28}$.

(b) Now we consider the Gregorian calendar system C_G. There are three factors determining the period of the sequence, the first two of which are the seven-day week and the four-year cycle of most leap years, which are the same as in (a). The third factor is due to the disruptions of the four-year cycle for leap years, which exhibit a period of 400 years. Hence, years p and q satisfy $C_G(p) = C_G(q)$ iff $p \equiv q \pmod 4$, $p \equiv q \pmod 7$, and $p \equiv q \pmod{400}$, which equivalently becomes $p \equiv q \pmod{2800}$. As a result, the cicada period of C_G is $\boxed{2800}$.

7. Let a, b, c be positive real numbers such that $a + b + c = \frac{3}{2}$. Show that

$$\frac{a}{a(2a^2+1)+b+c} + \frac{b}{b(2b^2+1)+c+a} + \frac{c}{c(2c^2+1)+a+b} \leq \frac{6}{7}.$$

SOLUTION. One finds that $\frac{x}{4x^3+3} \leq \frac{8x+3}{49}$ for all positive real x as

$$(8x+3)(4x^3+3) - 49x = (2x-1)^2(8x^2+11x+9) \geq 0.$$

Hence, letting $\sum_{\text{cyc}} f(a,b,c)$ denote the sum $f(a,b,c) + f(b,c,a) + f(c,a,b)$ for any function $f : \mathbb{R}^3 \to \mathbb{R}$, we obtain that

$$\sum_{\text{cyc}} \frac{a}{a(2a^2+1)+b+c} = \sum_{\text{cyc}} \frac{2a}{4a^3+3} \leq \frac{2}{49} \sum_{\text{cyc}} (8a+3) = \frac{6}{7},$$

as desired.

As a remark, to see why the linearization $\frac{x}{4x^3+3} \leq \frac{8x+3}{49}$ was used, try graphing both the function $f(x) = \frac{x}{4x^3+3}$ and the line $y = \frac{8x+3}{49}$ on a calculator, and see how they are related.