Berkeley Math Circle: Monthly Contest 7 Solutions

1. Let $ABCDE$ be a pentagon such the perpendicular bisector of AB contains the vertex D and intersects AB at a point P. Also suppose the pentagon is symmetric with respect to PD and that $AE \parallel BC$. If $BC = 6, AB = 14$, and $DP = 13$, what is the area of the pentagon?

   SOLUTION. By properties of parallel lines and transversals, we know $\angle EAB + \angle CBA = 180$. By the symmetry across DP, these two angles must have the same measure, so $\angle EAB = \angle CBA = 90$, and the quadrilateral $ABCE$ is a rectangle, so $CE = AB = 14$. If we let F be the intersection of DP and CE, then $FP \parallel CB$, so $DF = DP - FP = 13 - 6 = 7$. Hence, the area of $ABCDE$ equals the area of the rectangle $ABCE$ plus the area of the triangle CDE, which equals $(14)(6) + \frac{(14)(7)}{2} = 133$.

2. Suppose $|ab - cd| = |ac - bd| = |bc - ad|$. Prove that at least three of $|a|, |b|, |c|, |d|$ are equal.

   SOLUTION. Squaring, we have $(ab - cd)^2 = (ac - bd)^2$. Adding $2abcd$ to both sides gives $a^2b^2 + c^2d^2 = a^2c^2 + b^2d^2$, i.e. $(a^2 - d^2)(b^2 - c^2) = 0$. Thus, either $|a| = |d|$ or $|b| = |c|$. Likewise, either $|a| = |c|$ or $|b| = |d|$. Any combination must give the desired outcome.

3. Marquis and Sofiya are playing a game with n coins on a table. They take turns removing either 2, 5, or 6 coins at a time from the table. Once one of them can no longer remove either 2, 5, or 6 coins from the table, that person loses. If Marquis plays first, for what values of n does she have a winning strategy?

   SOLUTION. We show that the set of losing positions for Marquis is whenever the number of coins is congruent to 0, 1, 4, or 8 modulo 11. If Marquis is in a losing position and removes 5 or 6 coins, Sofiya can remove 6 or 5 coins, respectively, to return Marquis to a losing position. This method only becomes impossible when there are not enough coins left for Sofiya to remove to send Marquis to a losing position. For this to happen, there must be fewer than 11 coins left on Marquis' turn. The only losing position with at least 6 but fewer than 11 coins is 8 coins. From there, if Marquis removes either 5 or 6 coins, Sofiya can remove 2 coins to win.

   If Marquis instead chooses to remove 2 coins, there are several cases: if his position was congruent to 0 modulo 11 before removing 2, Sofiya can respond by removing 5. If his position was 1 modulo 11, Sofiya can respond by removing 6. If his position was 4 modulo 11, Sofiya can remove 2. If his position was 8 modulo 11, Sofiya can remove 5. All of these moves return Marquis to a losing position. So, Sofiya has a winning strategy whenever the number of coins n is congruent to 0, 1, 4, or 8 modulo 11.

   If n is congruent to 2, 3, 5, 6, 7, 9, or 10 modulo 11, then Marquis can remove 2, 2, 5, 6, 6, 5, or 6 coins, respectively, such that it is now Sofiya's turn and the number of coins remaining is congruent to 0, 1, 4, or 8 modulo 11. By repeating the argument and switching Marquis' and Sofiya's turn orders, this results in a winning position for Marquis.

4. If B, M, and C satisfy the equations

$$\begin{aligned} B+M+C &= 3, \\ \frac{1}{B}+\frac{1}{M}+\frac{1}{C} &= 4, \\ B^2+M^2+C^2 &= 5, \end{aligned}$$

what is the product BMC?

SOLUTION. Squaring the first equation give

$$(B+M+C)^2 B^2+M^2+C^2+2BM+2MC+2CB=9.$$

Subtracting the third equation from this one and then dividing by 2 gives $BM+MC+CB=2$. The second equation can then be rewritten as

$$\frac{BM+MC+CB}{BMC}=\frac{2}{BMC}=4.$$

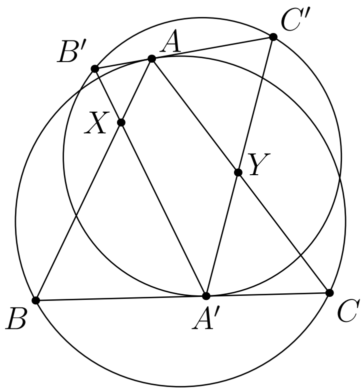
Solving for BMC we get $BMC=\boxed{\frac{1}{2}}$.

5. The theater club is putting on a play and needs to assign its n actors to the n roles. Each student signs up for two roles and each role is signed up for by two students. If every role must be given to a student who signed up for it, show that the number of ways the club can assign roles is a power of 2.

SOLUTION. Consider the multigraph where vertices are actors and edges are given by roles, where each edge corresponds to the actors who sign up for a specific role. As each vertex has degree 2, this graph is a disjoint union of cycles. Assign each cycle an arbitrary direction. Then in each cycle, either all actors are assigned to their incoming edge or their outgoing edge, making two possibilities. Then the number of possible ways to assign roles is 2^k for k the number of cycles, as desired.

6. Triangles ABC and $A'B'C'$ are such the circumcircle of ABC is tangent to $B'C'$ at A and the circumcircle of $A'B'C'$ is tangent to BC at A'. Let X be the intersection of AB and $A'B'$ and let Y be the intersection of AC and $A'C'$. Show that angle AXA' is equal to angle AYA'.

SOLUTION.



By the exterior angle theorem, $\angle AXA' = \angle XBA' + \angle XA'B = \angle ABC + \angle B'A'B$. Then by tangent angles, this is $\angle ABC + \angle B'C'A$. By symmetry, $\angle AYA'$ is also equal to this, as desired.

7. Show that there's a set S of integers such that every integer n is uniquely expressible as $20x + 23y$ for $x, y \in S$.

SOLUTION. We provide an algorithm to produce such a set. First begin with $T = \{0\}$. Write out the nonzero integers in an order, e.g. $1, -1, 2, -2, 3, -3, \ldots$. For each integer n, if n is already representable, move on, but otherwise, we will add elements to T to represent n.

By Bezout's theorem, one can pick integers a, b such that $20a + 23b = n$. Let N be a number at least a million times as large as a, b or any number in S in magnitude. Then add $X = a + 23N$ and $Y = b - 20N$ to T so that $20X + 23Y = n$. This makes n representable, while keeping all numbers already represented uniquely represented as X and Y are too large magnitude to represent anything already represented except for n, which by definition isn't represented already.

Let S be the limit of this algorithm, i.e. all numbers that eventually get added to T. By construction, S then works.