Berkeley Math Circle: Monthly Contest 4 Solutions

1. Find the smallest prime number with digit sum n or show that none exists, if n is:
   (a) 13.
   (b) 14.
   (c) 15.

   SOLUTION.

   (a) The first few numbers with digit sum 13 are 49, 58, and 67, so the first one that is prime is $\boxed{67}$.

   (b) The smallest number with digit sum 14 is 59, which is prime, so $\boxed{59}$ is the answer.

   (c) Since 15 is a multiple of 3, any number with digit sum 15 is a multiple of 3 and therefore not prime, so there are $\boxed{\text{no solutions}}$.

2. A solid is formed by gluing a prism to a pyramid so that the base of the pyramid lines up completely with one base of the prism. Is it possible for the solid to have exactly 2023:
   (a) faces?
   (b) vertices?
   (c) edges?

   SOLUTION. Suppose the common base of the pyramid and prism is an n-gon.

   (a) the solid has n faces from the pyramid, n vertical faces from the prism, ad the bottom face of the prism, for a total of $2n + 1$ faces. Thus, if $n = 1011$, there will be 2023 faces, so this is $\boxed{\text{possible}}$.

   (b) The solid has the $2n$ vertices from the prism, plus one extra vertex from the pyramid, for $2n + 1$ vertices. Again, this is $\boxed{\text{possible}}$ if we set $n = 1011$.

   (c) The solid has n edges from the bottom face of the prism, n vertical edges from the prism, n edges from the face shared between the prism and pyramid, and n more edges from the pyramid, for a total of $4n$ edges. Since 2023 is not a multiple of 4, this is $\boxed{\text{not possible}}$.

3. Marquis owns a factory where he produces bottled orange juice. He is preparing a shipment of 1,000,000 bottles of orange juice to send to his stores when he learns that a mistake in the factory caused one of the bottles to be poisoned! No one knows which bottle it might be, but Marquis has 20 poison testing devices to test the bottles. Each tester can only be used once, and he can administer as much orange juice as he wants to each of them. How can Marquis use the testers to determine which bottle is poisoned?

SOLUTION. Marquis can label each of the bottles 1 to 1,000,000 in binary. He will need 20 binary digits places to do so, and he can use a single tester to test all of the bottles with a 1 in the binary units place. He can use another tester to test all of the bottles with a 1 in the binary 2's place, another tester for the bottles with a 1 in the binary 4's place, and so on until all the bottles have been tested. The result of each test will tell whether the poisoned bottle has a 1 or a 0 in each binary digits place, so Marquis can determine which bottle was poisoned.

4. A polynomial of degree 2022 satisfies $P(n) = 2022^n$ for $n = 0, 1, \ldots, 2022$. What is $P(2023)$?

   SOLUTION. Since we are given 2023 data points for a degree 2022 polynomial, the polynomial must be uniquely determined, and writing $2022 = (1 + 2021)$, by the binomial theorem, it must be

$$P(n) = 2021^0 \binom{n}{0} + 2021^1 \binom{n}{1} + \cdots + 2021^{2022} \binom{n}{2022}.$$

   Plugging in $n = 2023$ gives $P(23) = \boxed{2022^{2023} - 2021^{2023}}$.

5. Aerith and Bob are playing a game on the edges of an infinite square grid. They alternate turns coloring one uncolored edge amber or bronze, respectively. Aerith wins if she forms a cycle, and Bob wins if he can prevent this indefinitely. Does Aerith have a winning strategy?

   SOLUTION. Bob wins. For every lattice point, Bob can ensure that he takes either the north or east edge of it. Then, Aerith cannot form a cycle as a bottom-left corner on the cycle would have at least one edge missing.

6. Show that no primes $2 < p < q < r$ exist so that $\frac{r-q}{p}$ and $\frac{r-p}{q}$ are both perfect squares.

   SOLUTION. Multiplying the former by p^2 and the latter by q^2, we see that $p(r-q)$ and $q(r-p)$ are both perfect squares. Let $m = \sqrt{p(r-q)}$ and $n = \sqrt{q(r-p)}$.

   By difference of squares, we have $(n+m)(n-m) = n^2 - m^2 = q(r-p) - p(r-q) = (qr - qp) - (pr - pq) = qr - pr = (q-p)r$. Thus, r either divides $(n+m)$ or $(n-m)$, so as $m, n \leq \sqrt{r^2} = r$, we must have $n + m = r$, so m and n have different parities, a contradiction as they are both even.

7. Andy the ant is crawling along the edges of a 4×4 grid. At each step, he chooses one of the horizontally or vertically adjacent vertices to crawl to with equal probability. For each starting point, what is the probability he reaches the bottom left corner A before the top right corner B?

SOLUTION. We will write each probability on top of the corresponding vertex. By symmetry, the set of probabilities must be as shown below for some values of p, q, and r:

$$\begin{array}{cccc} \frac{1}{2} & 1-q & 1-p & 0 \\ \\ q & \frac{1}{2} & 1-r & 1-p \\ \\ p & r & \frac{1}{2} & 1-q \\ \\ 1 & p & q & \frac{1}{2}. \end{array}$$

From each of the vertices labeled q, there is a $\frac{1}{3}$ chance of going to each of its three neighbors, and the probabilities of ending at A from those neighbors are p, $\frac{1}{2}$, and $\frac{1}{2}$, so the probability of ending at A starting from that vertex is

$$q = \frac{1}{3}\left(p + \frac{1}{2} + \frac{1}{2}\right) = \frac{1}{3}p + \frac{1}{3}. \tag{1}$$

Similarly, from the vertex labeled r, there is a $\frac{1}{4}$ chance of going to each of its neighbors, two of which have a p chance of ending at A, and two of which have a $\frac{1}{2}$ chance of ending at A. Thus,

$$r = \frac{1}{4}\left(p + p + \frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}p + \frac{1}{4}. \tag{2}$$

Finally, starting from each vertex labeled p, there is a $\frac{1}{3}$ chance of going to each of its three neighbors, and those neighbors have probabilities $1, r$, and q of ending up at A. Thus,

$$p = \frac{1}{3}\left(1 + r + q\right). \tag{3}$$

Plugging (1) and (2) into (3), we get

$$p = \frac{1}{3}\left(1 + \frac{1}{2}p + \frac{1}{4} + \frac{1}{3}p + \frac{1}{3}\right) = \frac{1}{3}\left(\frac{5}{6}p + \frac{19}{12}\right).$$

Solving for p gives $p = \frac{19}{26}$, and plugging this back into (1) and (2) gives

$$q = \frac{1}{3}\left(\frac{19}{2} + 1\right) = \frac{1}{3}\cdot\frac{45}{26} = \frac{15}{26}, \qquad r = \frac{1}{2}\left(\frac{19}{26} + \frac{1}{2}\right) = \frac{1}{2}\cdot\frac{32}{26} = \frac{8}{13}.$$

Plugging these value back into our table, we get

$$\boxed{\begin{array}{cccc} \frac{1}{2} & \frac{11}{26} & \frac{7}{26} & 0 \\ \\ \frac{15}{26} & \frac{1}{2} & \frac{5}{13} & \frac{7}{26} \\ \\ \frac{19}{26} & \frac{8}{13} & \frac{1}{2} & \frac{11}{26} \\ \\ 1 & \frac{19}{26} & \frac{15}{26} & \frac{1}{2}. \end{array}}$$