Berkeley Math Circle: Monthly Contest 3 Solutions

1. How many ways are there to write all numbers from 1 to 9 in the cells of a 3×3 grid so that for all integers $1 \leq n < 9$, the cell labeled n shares an edge with the cell labeled $n+1$?

   SOLUTION. Apply a checkerboard coloring to the grid so that the corners and the center are black. Then by parity, the 5 odd numbers must be in the 5 black cells and the 4 even numbers in the 4 white cells.

   If the center cell is "1", there are 4 ways to pick "2" and 2 ways to pick "3", at which point the rest of the numbers are determined, for a total of 8 possibilities. By symmetry there are 8 possibilities for the center cell to be 9. Likewise, if the center cell is "3", picking "2" and "1" gives 8 more, and another 8 more for the center cell being "7". Finally, if the center cell is "5", the path is still determined by picking "6" and "7". In total, there are therefore $5 \cdot 8 = 40$ possibilities.

2. A trapezoid has height 12 and diagonals of length 13 and 15. What is its area?

   SOLUTION. Place trapezoid $ABCD$ in the coordinate plane so that $A = (a, 12)$, $B = (b, 12)$. If $AC = 13$, by the Pythagorean theorem we then have $C = (a+5, 0)$, $D = (b-9, 0)$. Thus, the average length of the bases is $\frac{(b-a)+((a+5)-(b-9))}{2} = 7$, so the area of $ABCD$ is $7 \cdot 12 = 84$.

3. A sequence that starts with a positive number has the property that each of the following terms is the perimeter of the square with area equal to the preceding term. If the first three terms form an arithmetic sequence, what are the possible values for the first term of the sequence? (Having a common difference of 0 is allowed.)

   SOLUTION. Let the first term of the sequence be a. If a is the area of a square, then the side length of that square must be $\sqrt{a}$, so the second term must be $4\sqrt{a}$. Similarly, the third term must be $4\sqrt{4\sqrt{a}} = 8\sqrt[4]{a}$. If these terms form an arithmetic sequence, then they have a common difference so that

   $$4\sqrt{a} - a = 8\sqrt[4]{a} - 4\sqrt{a}.$$

   Letting $x = \sqrt[4]{a}$ gives

   $$x^4 - 8x^2 + 8x = 0,$$

   and since $a \neq 0 \implies \sqrt[4]{a} = x \neq 0$, we have

   $$x^3 - 8x + 8 = 0.$$

   We can observe that 2 is a solution to this equation, so we can finish by determining the solutions to $\frac{x^3-8x+8}{x-2} = x^2 + 2x - 4 = 0$. The only positive solution is $\sqrt{5} - 1$, but if $\sqrt[4]{a} = \sqrt{5} - 1$, then a would not be an integer. Hence, the only possible values for a is 16, in which case all three terms of the sequence are 16.

4. Let a be any positive integer. Show that there is always a Fibonacci number divisible by a.

   SOLUTION. Take the pair of values (F_i, F_{i+1}) modulo a. There can be no more than a^2 unique such pairs, so if we take these Fibonacci pairs up to (F_{a^2}, F_{a^2+1}), there must be two pairs which coincide modulo a, say they are (F_a, F_{a+1}) and (F_b, F_{b+1}) with $a < b$. But by the recursive definition of the Fibonacci numbers, we see that the pairs $(F_{a-1}, F_a), (F_{b-1}, F_b)$ must also coincide modulo a. We can continue this reduction until the pairs $(F_1, F_2) = (1, 1), (F_k, F_{k+1})$ coincide modulo a for some $k > 1$. Hence, $F_k \cong F_{k+1} \cong 1 \pmod a$, so $F_{k+1} - F_k = F_{k-1} \cong 0 \pmod a$.

5. Let S be a finite set of positive real numbers. If S's average is at most 1 but its product is at least 0.9, show that any three elements of S can form the sides of a triangle.

   SOLUTION. Assume otherwise for the sake of contradiction, i.e. that there exist $x, y, z \in S$ so that $x + y \leq z$. For fixed z, the product xy is then maximized when $x = y$. Then, if x, y, z have average a, their product is at most that when $x + y = z$, which happens at $x = 3a/4$, $y = 3a/4$, $z = 3a/2$, for a product of $27a^3/32$.

   Now consider replacing all of x, y, z with a in S. The product multiplies by at least $32/27$ for a product of at least $0.9 \cdot 32/27 > 1$, while the average is unchanged, a contradiction to AM-GM.

6. Show that the product of any two side lengths of a triangle is greater than the product of the diameters of the inscribed and circumscribed circles.

   SOLUTION. Let our triangle the ABC, a, b, c be the side lengths and r, R be the diameters of the circumscribed and inscribed circles, respectively. We want to show $ab > 4Rr$. The triangle inequality tells us that $a + b > c$. Heron's Formula tells us that the area of the triangle is $S = sr$ where $s = \frac{a+b+c}{2}$. The expression ab also occurs in another expression for the area, $S = \frac{ab\sin(C)}{2}$. Hence,

$$ab = 2S/\sin(C) = 2rs/(c/2R) = 2Rr(a + b + c)/c > 2Rr(c + c)/c = 4Rr.$$

7. Mathlandia has 2022 cities. Show that the number of ways to construct 2021 roads connecting pairs of cities such that it is possible to get between any two cities, there are no loops, and each city has exactly one or three roads coming out of it is given by

$$\frac{2022! \cdot 2019!!}{1012!}.$$

   (The notation 2019!! means $2019 \cdot 2017 \cdot \cdots \cdot 3 \cdot 1$.)

SOLUTION. If we consider the cities to be vertices and the roads to be edges, this arrangement is a type of graph known as a *trivalent tree*. We will find a general formula in terms of n for the number of trivalent trees on $2n$ vertices.

We can construct such a tree as follows. First, we will choose the internal vertices (non-leaves) of T. If there are i internal vertices, then summing the degrees over all vertices we get

$$2 \cdot (\# \text{ of edges}) = 2 \cdot (2n-1) = i \cdot 3 + (2n-i) \cdot 1 = 2i + 2n.$$

Solving for i gives $i = n-1$, so there are $\binom{2n}{n-1}$ ways to choose the internal vertices.

Now imagine attaching two edges to each of these $n-1$ vertices (with the other endpoint of each edge not yet chosen). This is $2n-2$ edges total, so all but one edge of the graph. We can think of the other endpoint of each edge as a position that needs to be filled, so at the start there are $2n-2$ available positions. Now, go through the leaves in increasing order, and for each leaf, choose an edge to attach it to. The number of available edges starts at $2n-2$ and goes down by one at each step as one position gets taken, so the number of ways to do this is $(2n-2)(2n-3)\dots(n-2)$.

After all the leaves are added, there must be at least one internal vertex both of whose edges are filled, since there are $n-1$ internal vertices and only $n-3$ unfilled edges left. Find the smallest such vertex, and choose an open edge to attach it to, which can be done in $n-3$ ways. Now that vertex has all three of its neighbors chosen.

After that, there are $n-2$ remaining internal vertices and $n-4$ available edges, all of which are attached to one of those vertices. So again, there must be a vertex both of whose edges are filled, so we can take the smallest such vertex and choose any of the $n-4$ edges to attach it to. We can continue this process until every internal vertex except for two of them has three edges, and at that point, we must add the final edge between those two vertices.

After this process is complete, we have a trivalent tree, but each such tree has been counted 2^{n-1} times, because we considered the two edges coming out of each internal vertex to be distinct, when actually they should not be. Thus, we must divide by 2^{n-1}, implying that the total number of trivalent trees is

$$\binom{2n}{n-1} \cdot \frac{(2n-2)(2n-3)\cdot \ldots \cdot 2 \cdot 1}{2^{n-1}} = \frac{(2n)!}{(n+1)!}(2n-3)!!.$$

Plugging in $n = 1011$ gives the desired expression.