Berkeley Math Circle: Monthly Contest 2 Solutions

1. Suppose that m is a two-digit positive integer. Let us form a new six-digit integer n by appending the two digits of m in the end of m twice. For example, if m is equal to 73, then n is 737373. Find four pairwise distinct prime numbers p, q, r, and s, all independent of the choice of m such that $pqrs \mid n$.

   SOLUTION. Observe that $n = 10000m + 100m + m = 10101m$, so that $10101 \mid n$ for all values of m. One may quickly find by trial and error that $10101 = 3 \cdot 7 \cdot 11 \cdot 37$, so setting $\{p, q, r, s\} = \boxed{\{3, 7, 11, 13\}}$ works.

2. Let $ABCD$ be a quadrilateral with $AC \perp BD$, and define E to be the intersection of diagonals AC and BD. Let A', B', C', and D' be the projections of E onto sides AB, BC, CD, and DA, respectively. Prove that $A'B'C'D'$ is cyclic.

   SOLUTION. By definition of projections,

   $\angle AA'E = \angle AD'E = \angle BB'E = \angle BA'E = \angle CC'E = \angle CB'E = \angle DD'E = \angle DC'E = 90^\circ$,

   so $A'BB'E$, $B'CC'E$, $C'DD'E$, and $D'AA'E$ are all cyclic. Then

   $$\angle A'B'C' = \angle A'B'E + \angle C'B'E = \angle A'BE + \angle ECC'.$$

   Since $AC \perp BD$, so that $\triangle AEB$ and $\triangle CED$ are both right triangles, we have

   $$\begin{aligned}\angle A'BE + \angle ECC' &= (90^\circ - \angle A'AE) + (90^\circ - \angle C'DE) \\ &= 180^\circ - (\angle A'AE + \angle C'DE) \\ &= 180^\circ - (\angle A'D'E + \angle C'D'E) \\ &= 180^\circ - \angle A'D'C'.\end{aligned}$$

   Hence

   $$\angle A'B'C' + \angle A'D'C' = 180^\circ,$$

   which implies that $A'B'C'D'$ is cyclic.

3. Aerith and Bob take turns naming prime numbers. Aerith starts with the number 2. Each turn, a player must add a positive integer to the current number to get another prime number, adding no more than twice what is necessary to do so. In other words, if the current number is n and the next prime number is p, the player may not add more than $2(p - n)$.

   Both players want to be the one who says 4567. Who has a winning strategy?

   SOLUTION. $\boxed{\text{Aerith}}$ does. After Bob says 3, Aerith has the choice to say either 5 or 7. If saying 7 were a winning strategy, she would say it. But if saying 7 is losing, she can force Bob to say it by saying 5.

4. Let $p(x) = x^3 - ax^2 + bx - c$ be a polynomial for some positive real numbers a, b, and c. Prove that if $p(x)$ has three positive roots, counting multiplicity,

$$a^2b^2 \geq 2b^3 + 27c^2.$$

SOLUTION. It suffices to prove that the above inequality must be true when $p(x)$ has three positive roots x_1, x_2, and x_3. Since the x_i are positive, the QM-HM Inequality gives

$$\begin{aligned}\sqrt{\frac{(x_1+x_2+x_3)^2 - 2(x_1x_2+x_2x_3+x_3x_1)}{3}} &= \sqrt{\frac{x_1^2+x_2^2+x_3^2}{3}} \\ &\geq \frac{3}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}} = \frac{3x_1x_2x_3}{x_1x_2+x_2x_3+x_3x_1}.\end{aligned}$$

Then substituting Vieta's Formulas

$$(a, b, c) = (x_1+x_2+x_3, x_1x_2+x_2x_3+x_3x_1, x_1x_2x_3)$$

into the above inequality implies that

$$\sqrt{\frac{a^2-2b}{3}} \geq \frac{3c}{b},$$

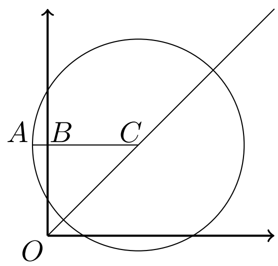
which rearranges to

$$a^2b^2 \geq 2b^3 + 27c^2,$$

as desired.

5. The center of a circle lies on the bisector of quadrant I of the coordinate system. The distance between the center of the circle and the origin of the coordinate system is d, and the radius of the circle is r, where both r and d are independent random variables uniformly distributed in $[0, 1]$. Find the probability that part of the circle lies in quadrants II and IV, but not in quadrant III.

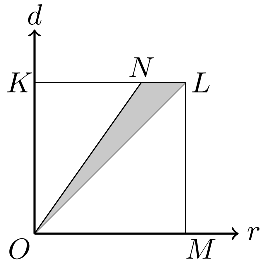
SOLUTION. Let C be the center of the circle and let AC is the radius perpendicular to the y-axis. Set $OC = d$ and $AC = r$; it quickly follows that $BC = \frac{d}{\sqrt{2}}$.



Now it follows that our desired condition that some part of the circle lies nontrivially in quadrants II and IV but not III occurs iff

$$\frac{d}{\sqrt{2}} < r < d.$$

Since r and are chosen uniformly and independently randomly over the range $[0, 1]$, we can interpret (r, d) as a random point uniformly chosen in the unit square $OKLM$, as shown below.



Then the solution to the original two-sided inequality $\frac{d}{\sqrt{2}} < r < d$ is represented by exactly the points inside the gray triangle $\triangle ONL$, which is bounded between the lines $r = d$, $\sqrt{2}r = d$, and $d = 1$. Then $OK = 1$ and $NL = 1 - \frac{1}{\sqrt{2}}$, so it follows that our desired probability is

$$\frac{[ONL]}{[OKLM]} = \frac{NL \cdot KO}{2} = \boxed{\frac{2-\sqrt{2}}{4}}.$$

6. For which real number a_0 does the recursive sequence defined by $a_n = 1 + na_{n-1}$ converge to zero?

SOLUTION. The answer is $\boxed{1-e}$, as one can verify that this value of a_0 gives

$$a_n = -\sum_{m>n} \frac{n!}{m!} = -\sum_{m>n} \frac{1}{m!/n!},$$

which indeed converges to zero.

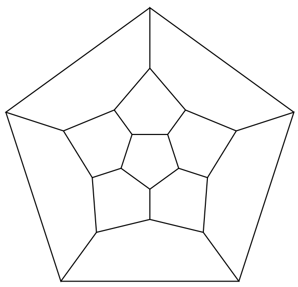
7. Consider an assignment of integers to each edge of a regular dodecahedron, so that the following criteria are met.

   a) At each vertex, the sum of the numbers on the edges leading to that vertex is a multiple of 3.

   b) At each vertex, the product of two of the numbers on the edges leading to that vertex leaves a remainder of 2 when divided by 3.

   Prove that there are exactly 3 pairs of opposite edges whose numbers leave the same remainder when divided by 3, and show that each of these 3 remainders are distinct.

   The diagram shown below is a planar representation of a solid dodecahedron.



SOLUTION. We consider all edge numbers modulo 3. To satisfy part (a), a quick inspection of the possible residue classes shows that there are only 2 possibilities for the residue classes of 3 edges meeting at a point, namely:

a) $0, 1, 2$.

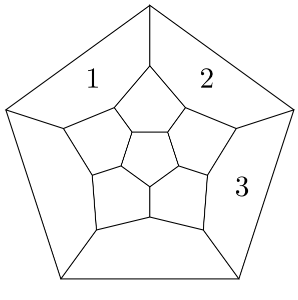
b) x, x, x, where x is some residue class modulo 3.

The second possibility fails because the quadratic residues (residue classes corresponding to perfect squares) modulo 3 are 0 and 1.

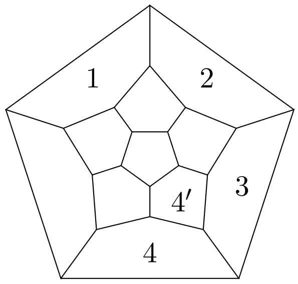
We now consider a path on the dodecahedron defined as follows. The starting vertex, v_0, is arbitrarily chosen, and the second vertex v_1 is chosen so that the edge joining v_0 and v_1 has value 1. For any vertex v_k, the successor v_{k+1} is chosen as the unique vertex adjacent to v_k so that the edge joining v_k and v_{k+1} has value $(-1)^k$. The successor defines a permutation on the vertices of the dodecahedron because it has an inverse, defined by $v_k \mapsto w$ where w is the unique vertex so that the edge joining v_k to w has value $(-1)^{k+1}$. To see that it defines a permutation on the *entire* set of vertices, we simply apply the same principle to any remaining vertices that were not covered in the initial cycle. Note that for a cycle of length n defined by the above procedure, we must have $(-1)^0 = (-1)^n$ and therefore n is even.

Any permutation can be decomposed into a disjoint union of cycles; here, this just says that we can partition the set of vertices into some cycles of even length. Any such cycle on a dodecahedron divides it into two regions, each composed of some number of pentagons. We claim that we may assume without loss of generality that one of these regions consists of some pentagons, say p of them, arranged in a linear fashion. To see this, assume for contradiction that this is not the case, in which case we must have at least one tripe of pentagons in one of these regions that meet at a point, and then pass to the cycle containing this point. We can now freely count the number of edges on the perimeter of such a linear-pentagonal-like shape as follows. We start with 5 edges, and for each of the other $p-1$ pentagons we add, we add 3 new edges to our figure, thus bringing the perimeter to $n = 2 + 3p$ edges. Recall that n is even, and therefore so is p. We can now read off the possible values of p as 2, 4, or 6, which correspond to $n = 8, 14$, or 20. We need a total of 20 edges, so we just need to write 20 as a sum of some 8's, some 14's, and some 20's. The only way to do this is with one 20. Of course, we have to be a little bit more careful because we only know that *one* of the cycles can be expressed as the perimeter of such a shape; it cannot be of length 14 because that would leave only 6 vertices remaining, and the smallest possible cycle has length 8, so we would conclude that all cycles have length 8, a contradiction. Thus, we have 6 pentagons on one side, arranged in a linear fashion, with a total perimeter of 20 edges, which pass through every vertex.

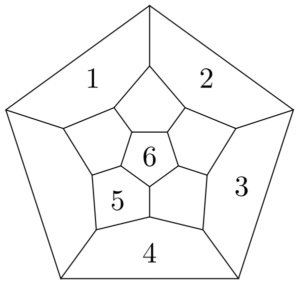
We now proceed to construct the unique (up to symmetry) cycle. The first three pentagons can be chosen to be the following, by rotational symmetry.



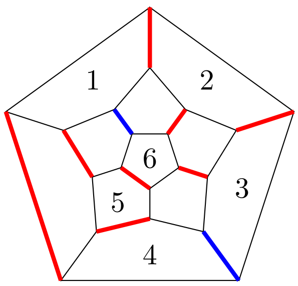
The fourth pentagon can be chose to be either of the following:



However, note that in the case marked with a ′, there is only one pentagon left to be chosen (assuming 1 marks the starting pentagon), so we proceed with the case marked without a ′. It follows from a brief inspection that the only possibility is as follows:



We are now free to draw all the edges that have a zero value (the ones not on the perimeter).



The ones drawn in blue are the only opposite pair, and so we have shown that there is exactly one pair of opposite edges that are congruent to 0 modulo 3. We can then add 1 to all the edge numbers, yielding the desired result for the residue classes of 1 and 2, completing our proof.