# Berkeley Math Circle: Monthly Contest 1 Solutions

1. If integers $a, b, c, d$ are such that $a + b = c + d$, prove that $abcd + 1$ is not a multiple of three.

   **SOLUTION.** If any of $abcd$ are divisible by 3, this is clear. Otherwise, all numbers are 1 or 2 modulo 3, so all possible sums $a+b$ are $1+1$, $1+2$, and $2+2$. These have distinct residues modulo 3, so $\{a, b\} \equiv \{c, d\} \pmod 3$. Thus, $ab \equiv cd \pmod 3$, so $abcd$ is a perfect square modulo 3 and thus cannot be 2, as desired.

2. Let $A, B, C, D, E$ be five points in the plane satisfying $AB = BC = CD = DE = EA$ and $AC = CE = EB = BD = DA$. Show that $ABCDE$ is a regular pentagon.

   **SOLUTION.** We have $y = \angle EAB = \angle ABC = \angle BCD = \angle CDE = \angle DEA$ by SSS congruence. Let $y = 180 - x$ for some $0 < x < 180$. Then the angle between vectors $AB$ and $BC$ is $x$, and the same holds cyclically. Thus $\pm x \pm x \pm x \pm x \pm x$ is a multiple of 360, so $x = \frac{360m}{n}$ for some odd $n \le 5$. Since $n$ cannot be 1, and one can check by hand that it cannot be 3, all the $\pm$ signs have the same orientation, giving the desired.

3. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying $f(x) + f(y) - f(x + y) = f(xy + 1)$ for all integers $x$ and $y$.

   **SOLUTION.** Setting $x = y = 0$, we instantly find that $f(0) + f(0) - f(0) = f(1)$, simplifying to $f(0) = f(1)$.

   Setting $x = 1$ and $y = -1$, we then find that $f(-1) + f(1) - f(0) = f(0)$. Since $f(0) = f(1)$, this implies that $f(-1) = f(0)$.

   Next, we prove by induction on $|m|$ that $f(m) = f(0)$ for all integers $m$.

   Our base case, with $|m| = 1$, has already been completed. Therefore, it suffices to show that if $f(m) = f(0)$ for all $|m| = n$, then $f(n + 1) = f(-n - 1) = f(0)$.

   Setting $x = -1$ and $y = n + 1$ in our original equation, we find that

   $$f(-1) + f(n + 1) - f(n) = f(-n).$$

   Because $f(n) = f(-n) = f(0)$ by the inductive hypothesis and we already know that $f(-1) = f(0)$, the above reduces to $f(n + 1) = f(0)$.

   Then, setting $x = 1$ and $y = -n - 1$ in our original equation, we find that

   $$f(1) + f(-n - 1) - f(-n) = f(-n).$$

   Again by the inductive hypothesis, we know that $f(-n) = f(0)$, so our equation reduces to $f(-n - 1) = f(1) = f(0)$. This completes the induction.

   Since $f(m) = f(0)$ for all integers $m$, it follows that $f$ must be the constant function $\boxed{f(x) = c}$ for any integer $c$, which can trivially be seen to work.

4. Jessica rolls a fair die until she gets three 6s in a row, at which point she stops. What is the expected number of times that Jessica will roll the die?

**SOLUTION.** Let $E$ be the expected number of rolls before getting three 6s. For each of Jessica's rolls, observe that exactly one of four mutually exclusive events will occur:

a) The first roll is not a 6, in which case Jessica has to start the process over and add the first roll to her total. This happens with probability $\frac{5}{6}$. Hence this event adds a value of $\frac{5}{6}(E+1)$ to the expected value.

b) The first roll is a 6 but the second is not, in which case Jessica has to start the process over and add the first two rolls to her total. Since this occurs with probability $\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)$, we thus add $\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)(E+2)$ to our expected value.

c) The first and second rolls are 6s, but the third is not, in which case Jessica has to start the process over and add the first three rolls to her total. This occurs with probability $\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$, adding $\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)(E+3)$ to our expected value.

d) The first three rolls are 6s, occurring with probability $\left(\frac{1}{6}\right)^3$, in which case Jessica stops right away, contributing $\left(\frac{1}{6}\right)^3 3$ to our expected value.

Putting everything together, we thus find that

$$E = \frac{5}{6}(E+1) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)(E+2) + \left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)(E+3) + \left(\frac{1}{6}\right)^3 3 = \frac{215E}{216} + \frac{43}{36}.$$

Solving for $E$ yields

$$E = \left(1 - \frac{215}{216}\right)\frac{43}{36} = \boxed{258}.$$

5. Evan Corporation LLC has 956 employees, 24 departments, and a total budget of \$215100. If V. Enhance, the company's CEO, sends $m$ employees and $n$ dollars to a certain department, then that department will generate a total revenue of $100\sqrt{mn}$ dollars annually. What is the maximum possible total revenue that Evan Corporation LLC can generate annually? Note that Mr. Enhance does not count as an employee.

**SOLUTION.** For some $i \in \{1, 2, \cdots, 24\}$, suppose that Mr. Enhance allocates $m_i$ employees and $n_i$ dollars to department $i$, so that $m_1 + m_2 + \cdots + m_{24} = 956$ and $n_1 + n_2 + \cdots + n_{24} = 215100$. Then department $i$ generates $100\sqrt{m_i n_i}$ dollars. Summing across all 24 departments, the total revenue $R$ is given by

$$R = 100(\sqrt{m_1 n_1} + \sqrt{m_2 n_2} + \cdots + \sqrt{m_{24} n_{24}}).$$

By the Cauchy-Schwarz Inequality, we have that

$$\begin{aligned}
&(\sqrt{m_1 n_1} + \sqrt{m_2 n_2} + \cdots + \sqrt{m_{24} n_{24}})^2 \\
&\leq (\sqrt{m_1}^2 + \sqrt{m_2}^2 + \cdots + \sqrt{m_{24}}^2)(\sqrt{n_1}^2 + \sqrt{n_2}^2 + \cdots + \sqrt{n_{24}}^2) \\
&= (m_1 + m_2 + \cdots + m_{24})(n_1 + n_2 + \cdots + n_{24}) \\
&= 956 \cdot 215100.
\end{aligned}$$

Since $956 = 4 \cdot 239$ and $215100 = 900 \cdot 239$, it follows that

$$\sqrt{m_1 n_1} + \sqrt{m_2 n_2} + \cdots + \sqrt{m_{24} n_{24}} \leq \sqrt{956 \cdot 215100} = \sqrt{(4 \cdot 239)(900 \cdot 239)} = 60 \cdot 239,$$

so that

$$R = 100(\sqrt{m_1 n_1} + \sqrt{m_2 n_2} + \cdots + \sqrt{m_{24} n_{24}}) \leq 100 \cdot 60 \cdot 239 = 1434000.$$

Due to the Cauchy-Schwarz Inequality, equality is achieved above iff $\frac{\sqrt{m_i}}{\sqrt{n_i}}$ and thus $\frac{m_i}{n_i}$ is constant for all $i$. Thus the maximum possible revenue generated is $\boxed{\$1434000}$.

6. A gambler starts with an initial amount of \$3 and makes a series of bets. The gambler earns \$1 upon winning a bet and loses \$1 otherwise. The gambler stops betting when he either reaches a total amount of \$6 or completely runs out of money. Letting $p$ be the probability of the gambler losing an arbitrary individual bet, independent of all other bets, compute, with proof, the probability that gambler eventually goes broke.

   **SOLUTION.** As the gambler makes a series of bets, his current amount of money moves by 1 up or down among the integers between 0 and 6, inclusive. Once the current amount reaches either 0 or 6, the series ends.

   Now, for any $0 \leq i \leq 6$, let $x_i$ be the probability that the series ends at 0 if the initial amount is $i$ and $q = 1 - p$ be the probability that the gambler wins a bet. If the gambler currently has $i$ dollars for some $1 \leq i \leq 5$, then the probability $x_i$ that the series of bets eventually ends at \$0 equals the sum of the probabilities of the following two events:

   a) The gambler's next bet is a loss, occurring with probability $p$, times the probability that the series starting at $i - 1$ ends at 0, with probability $x_{i-1}$, or

   b) The next bet is a win, occurring with probability $q$, times the probability that the series starting at $x + 1$ ends at 0, with probability $x_{i+1}$.

   It thus follows that

$$x_i = p x_{i-1} + q x_{i+1}.$$

   Since $x_0 = 1$ and $x_6 = 0$, we apply the above equation to get the following system of 5 linear equations:

$$\begin{aligned} x_1 &= p + q x_2, \\ x_2 &= p x_1 + q x_3, \\ x_3 &= p x_2 + q x_4, \\ x_4 &= p x_3 + q x_5, \\ x_5 &= p x_4. \end{aligned}$$

   It suffices to find $x_3$, so we substitute the value of $x_1$ from the first equation into the second one and also substitute the value of $x_2$ from the second equation into the third one. Similarly, we substitute the value of $x_5$ from the fifth equation into the fourth one and the value of $x_4$ from the fourth equation into the third one. One can check that the final result is

$$x_3 = \frac{p^3}{1 - 3pq} = \boxed{\frac{p^3}{3p^2 - 3p + 1}}.$$

7. A sequence of real numbers $a_n$ satisfies $\min(a_m, a_n) = a_{\gcd(m,n)}$ for all positive integers $m, n$. Must we have $a_1 + \cdots + a_{26} \le a_{2023} + \cdots + a_{2048}$?

**SOLUTION.** For a fixed real number $k$, consider the smallest $n_k$ such that $a_{n_k} \ge k$. If $a_m \ge k$, then $a_m \ge a_{n_k}$, for otherwise $a_m = \min(a_m, a_{n_k}) = a_{\gcd(m,n_k)}$, which is less than $k$ because $\gcd(m, n_k) \le n_k$ by definition of $n_k$, a contradiction. Thus

$$\begin{aligned} a_m \ge k &\Leftrightarrow a_{n_k} = \min(a_m, a_{n_k}) = a_{\gcd(m,n_k)}, \\ &\Leftrightarrow \gcd(m, n_k) \ge n_k, \\ &\Leftrightarrow \gcd(m, n_k) = n_k, \\ &\Leftrightarrow n_k \mid m. \end{aligned}$$

Now note that there are at most as many numbers in $\{a_1, \ldots, a_{26}\}$ that are at least $k$ as there are numbers in $\{a_{2023}, \ldots, a_{2048}\}$ at least $k$, for the number of multiples of $n_k$ in the former set is $\left\lfloor \frac{11}{n_k} \right\rfloor$, which is the minimum possible number of these multiplies in the latter.

Hence, due to the above results, if one sorts the numbers in the sum $a_{2023} + \cdots + a_{2048}$ in decreasing order, the $j$th term in the sorted sequence will always be at least the $j$th term in $a_1 + \cdots + a_{26}$. Therefore the inequality $a_1 + \cdots + a_{26} \le a_{2023} + \cdots + a_{2048}$ must always hold.