Berkeley Math Circle: Monthly Contest 5 Solutions

1. Find all possible values for the units digit of n^{2022}, where n is a positive integer.

   SOLUTION. The units digit of n^{2022} depends only on the units digit of n, so we can consider cases based on the units digit of n:

   a) If n has units digit 0, then n^{2022} has units digit 0.

   b) If n has units digit 1, then n^{2022} has units digit 1.

   c) If n has units digit 2, the units digits of the powers of n follow a cycle $2, 4, 8, 6, \dots$ of length 4. Since $2022 \equiv 2 \pmod 4$, the units digit of n^{2022} is the same as the units digit of n^2, which is 4.

   d) If n has units digit 3, the units digits of its powers follow a cycle $3, 9, 7, 1, \dots$ of length 4, so n^{2022} has the same units digit as n^2, which is 9.

   e) If n has units digit 4, the units digits of its powers alternate $4, 6, 4, 6, \dots$, so since 2022 is even, n^{2022} has units digit 6.

   f) If n has units digit 5, then n^{2022} has units digit 5.

   g) If n has units digit 6, then n^{2022} has units digit 6.

   h) If n has units digit 7, the units digits of its powers follow a cycle $7, 9, 3, 1, \dots$ of length 4, so n^{2022} has the same units digit as n^2, which is 9.

   i) If n has units digit 8, the units digits of its powers follow a cycle $8, 4, 2, 6, \dots$ of length 4, so n^{2022} has the same units digit as n^2, which is 4.

   j) If n has units digit 9, the units digits of its powers alternate $9, 1, 9, 1, \dots$, so since 2022 is even, n^{2022} has units digit 1.

   Putting this together, the possible units digits are $\boxed{0, 1, 4, 5, 6, 9}$.

2. Robert has two teenagers, both either male or female, with uniformly random but distinct ages from 13 to 19. Given that he has at least one daughter whose age is prime, what is the probability that both his children are female?

   SOLUTION. If he has a boy and a girl, there are 3 possibilities for the girl's age, and 6 remaining possibilities for the boy's, for a total of 18 possibilities.

   If he has two daughters, there are 3 ways for both their ages to be prime, and $3 \cdot 4 = 12$ ways for one's age to be prime and the other's composite.

   Thus, there are $3 + 12 + 18 = 33$ total possibilities, in $3 + 12 = 15$ of which Robert has two daughters. Thus the answer is $\frac{15}{33} = \frac{5}{11}$.

3. In the equation

$$\begin{array}{r} BMC \\ FUN \\ +\,MATH \\ \hline 2\;0\;2\;2 \end{array}$$

each letter represents a distinct digit, with no leading 0's. What is the largest possible value of BMC?

SOLUTION. First, notice that we must have $M = 1$, since $B + F + A > 0$, so a 1 gets carried from the 100's to 1000's place. There must also be a 1 (or a 2) carried from the 10's to 100's place, since we can't have $M + U + T = 2$ with M, U, and T all distinct. This means $B + F + A + 1 \leq 10$. To make B as large as possible, we should set $B = 7, F = 2, A = 0$.

Now to make BMC as large as possible, we should try to set $C = 9$. So we are now trying to solve:

$$\begin{array}{r} 7\,1\,9 \\ 2\,U\,N \\ +\,1\,0\,T\,H \\ \hline 2\,0\,2\,2 \end{array}$$

For $9+N+H$ to have 1's digit 2, we need $N+H = 3$ or $N+H = 13$. The remaining digits we can use are 3, 4, 5, 6, and 8, so the only possibility is for N and H to be 5 and 8 in some order. This means we carry a 2 to the 10's place, so we need $1 + U + T + 2 = 12$, meaning $U + T = 9$. The remaining digits we can use are 3, 4, and 6, so U and T must be 3 and 6 in some order. Thus, $BMC = 719$ is indeed possible, and in fact has 4 different solutions:

$$\begin{aligned} 719 + 268 + 1035 &= 2022, \\ 719 + 265 + 1038 &= 2022, \\ 719 + 238 + 1065 &= 2022, \\ 719 + 235 + 1068 &= 2022. \end{aligned}$$

Our answer is thus $BMC = \boxed{719}$.

4. Given four points in the plane that don't lie on a circle and no three of which lie on a line, how many circles are equidistant from all four points? (The distance from a point to a circle means the distance from the point to the closest point on the circle.)

SOLUTION. Since the four points do not lie on a circle, there is no circle equidistant from the four points such that all four points are inside it or all four points are outside it. We claim now that if we split the points in any way into either two pairs or a group of one and a group of three, then there is exactly one circle separating the two groups.

Suppose first we would like point A to be on one side of the circle and points B, C, and D to be on the other side. Then the center of the circle must be the circumcenter O of $\triangle BCD$. Now suppose the circumcircle of $\triangle BCD$ meets line AO at point E on the same side of O as A. Then the circle must pass through the midpoint M of AE, since then it has distance $MA = MA$ to all four points. Thus, the circle is uniquely determined.

Now suppose we would like points A and B to be on one side of the circle and points C and D to be on the other side. Then the center of the circle must be equidistant from A and B and also equidistant from C and D, forcing it to be the intersection of the perpendicular bisectors of AB and CD. From this point, the circle is uniquely determined by similar reasoning to the other case.

Putting this together, if we are given four points A, B, C, and D, then there are three ways to split them into two pairs (without regard to order), and four ways to split them into a group of one and a group of three. Each of these ways of splitting the points gives a unique circle, so there are $\boxed{7}$ circles total.

5. Find all positive integers n such that n^2+1 is a prime and $5n^2+1$ is a perfect square.

SOLUTION. Let $m = \sqrt{5n^2+1}$. We get $n^2+1 = m^2 - (2n)^2 = (m-2n)(m+2n)$. Since this is prime, $m - 2n$ is thus 1, so $m = 2n + 1$. Thus, $(2n + 1)^2 = 5n^2 + 1$, giving $4n = n^2$, or $n = 4$.

6. Anna the ant is crawling along the number line starts at 1, and for each step, she goes one unit to the left with probability p and one unit to the right with probability $1 - p$, where $p > \frac{1}{2}$. What is the expected number of steps it takes her to reach 0?

SOLUTION. Let x be the expected number of steps. It always takes at least 1 step. If the first step is to the right (which happens with probability $1 - p$), then it takes in expectation x steps to get from 2 to 1, and then another x steps to get from 1 to 0. Thus, we get

$$x = 1 + (1 - p)2x.$$

Rearranging gives $(2p - 1)x = 1$, so $x = \boxed{\dfrac{1}{2p-1}}$.

7. Find the number of one-to-one functions $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ satisfying the following two properties:

   (1) There do not exist $a < b < c$ such that $f(a) < f(b) < f(c)$.

   (2) There do not exist $a < b < c < d$ such that $f(b) < f(a) < f(d) < f(c)$.

SOLUTION. These are known as *pattern-avoiding permutations.* An *occurrence of a permutation pattern* means a subsequence of the permutation ordered in the same way as that pattern. If there are no such occurrences, then the permutation *avoids* the pattern. Condition (i) means that the permutation avoids the pattern 123, and condition (ii) means it avoids the pattern 2143. So we are looking for the number of permutations of length n with are 123-avoiding and 2143-avoiding.

We claim that the number of such permutations is the Fibonacci number F_{2n-1}. We will prove this by induction on n. For $n = 1$, the single permutation works, so the number of ways is $1 = F_1$, and for $n = 2$, both permutations work, so the number of ways is $2 = F_3$. This proves our base cases.

Now assume the statement holds for all $k < n$. for some $n \geq 3$. We will break this into cases based on the position of n in the one-line notation for the permutation. If n is first or second, then it can't be involved in any occurrence of 123 or 2143, so any 123-avoiding and 2143-avoiding of the remaining values works, and thus by the inductive hypothesis there there are $F_{2(n-1)-1} = F_{2n-3}$ ways in each case.

If n is in position k with $k \geq 3$, then the values in positions $1, 2, \ldots, k-1$ must be in decreasing order to avoid an occurrence of 123. Also, the first $k-2$ of them must be greater than all the values in position $k+1$ or later to avoid an occurrence of 2143. The only way this can happen is if the first $k-2$ values are $n-1, n-2, \ldots, n-k+2$ in that order. The remaining entries can be any 123-avoiding and 2143-avoiding permutation of $1, 2, \ldots, n-k+1$, so there are $F_{2(n-k+1)-1}$ ways in this case. Putting this together and repeatedly using the Fibonacci recurrence, we find that the number of ways for n is

$$\begin{aligned} F_1 + F_3 + F_5 + \cdots + F_{2n-5} + 2F_{2n-3} &= F_2 + F_3 + F_5 + F_7 + \cdots + F_{2n-5} + 2F_{2n-3} \\ &= F_4 + F_5 + F_7 + \cdots + F_{2n-5} + 2F_{2n-3} \\ &= F_6 + F_7 + \cdots + F_{2n-5} + 2F_{2n-3} \\ &\ldots \\ &= F_{2n-4} + 2F_{2n-3} = F_{2n-2} + F_{2n-3} = F_{2n-1}. \end{aligned}$$

This completes the induction, so the formula holds for all n.