Berkeley Math Circle: Monthly Contest 8 Solutions

1. If $a < b < c < d$ are distinct positive integers such that $a + b + c + d$ is a square, what is the minimum value of $c + d$?

   SOLUTION. The answer is 11, which comes from $2 + 3 + 5 + 6$ or $1 + 4 + 5 + 6$.

   The minimum possible value of $a + b + c + d$ for $a < b < c < d$ is $1 + 2 + 3 + 4 = 10$. The smallest square at least 10 is 4^2.

   Now assume for sake of contradiction that $c+d \leq 10$. Then, $c < \frac{c+d}{2} \leq 5$, so $a < b < c \leq 4$ and $a+b \leq 2+3 = 5$. Therefore, $16 \leq a+b+c+d = (a+b)+(c+d) \leq 5+10 = 15$, a contradiction, confirming that 11 is minimal.

2. An unfair coin comes up heads with probability $\frac{4}{7}$ and tails with probability $\frac{3}{7}$. Aerith and Bob take turns flipping the coin until one of them flips tails, with Aerith going first. What is the probability that Aerith wins the game?

   SOLUTION. Let p be the probability Aerith wins. There is a $\frac{3}{7}$ chance Aerith wins right away, and if she doesn't win right away, there is a $1 - p$ chance she will win after that (since at that point it's like starting the game over but with Bob going first. This gives
   $$p = \frac{3}{7} + \frac{4}{7}(1 - p) = 1 - \frac{4}{7}p,$$
   and solving this we get $p = \boxed{\frac{7}{11}}$.

3. Unit segments AB and CD are given such that $AB \nparallel CD$. If M is the midpoint of AC, N the midpoint of BD, and X the intersection of lines AB and CD, given $M \neq N$ and $A \neq X \neq C$, show that the angle bisector of $\angle AXC$ is either parallel or perpendicular to MN.

   SOLUTION. Let E be such that $ABED$ is a parallelogram. ED is then a unit segment parallel to AB, so the angle bisector of $\angle CDE$ is parallel either to the internal or to the external angle bisector of $\angle AXC$, the angle between lines AB and CD. Furthermore, the midpoint of BD, N, is also the midpoint of AE, so MN is the A-midline of triangle CE and MN is parallel to CE. Because CDE is isosceles, the angle bisector of $\angle CDE$ is the D-altitude perpendicular to CE. Combining the above observations gives that MN is perpendicular to the angle bisector of $\angle CDE$, which is either parallel or perpendicular to the angle bisector of $\angle AXC$, as desired.

4. Aerith picks five numbers and for every three of them, takes their product, producing ten products. She tells Bob that the nine smallest positive divisors of sixty are among her products. Can Bob figure out the last product?

SOLUTION. Yes Bob can. Let the numbers be v, w, x, y, z and let the missing product be xyz. The nine given products are $1, 2, 3, 4, 5, 6, 10, 12, 15$.

Partition these products into the three sequences

$$s_1 = (vyz, vzx, vxy), s_2 = (wyz, wzx, wxy), s_3 = (vwx, vwy, vwz)$$

and call these sequences *special*. Note that $s' = (yz, zx, xy)$ is directly proportional to the first two and inversely proportional to the last one. The multiples of 5 thus have to be in the same special sequence, WLOG $(5, 10, 15)$. This sequence is inversely proportional to at least one of other two special sequences, which would thus have to be a multiple of $(6, 3, 2)$. However, it cannot be $(6, 3, 2)$ itself as that would leave 1, 4, and 12, which cannot form a special sequence. Thus, the multiple of $(6, 3, 2)$ has to be $(12, 6, 4)$, leaving $(1, 2, 3)$ as the final sequence. WLOG, we thus can get the equations

$$((vyz, vzx, vxy), (wyz, wzx, wxy), (vwx, vwy, vwz)) = ((5, 10, 15), (1, 2, 3), (12, 6, 4)).$$

We can then deduce $xyz = vyz \cdot wxz/vwz = 5 \cdot 2/4 = 5/2$.

Indeed, $5/2$ is achieved by

$$(v, w, x, y, z) = \left(2\sqrt[3]{15}, \frac{2}{5}\sqrt[3]{15}, \sqrt[3]{15}, \frac{1}{2}\sqrt[3]{15}, \frac{1}{3}\sqrt[3]{15}\right).$$

5. Aerith writes a positive integer in each cell of a 2021×2021 grid. Every second, Bob will pick a cell with value at least five if one exists, decrease its value by four, and increment each of the cell's neighbors by 1. Must this process always stop?

   SOLUTION. Yes. Let the sum of all numbers in the grid be S. Note that S never increases, as each cell has at most 4 neighbors; in fact, it decreases when Bob picks an cell on the boundary. There are finitely many grids with total sum $\leq S$, so it suffices to show that Bob cannot encounter the same grid twice. Assume for sake of contradiction that he could indeed loop through some sequence of operations; note that he would have to preserve S the whole time.

   The numbers in the top row can never change because if they did, they must at some point decrease in the loop, which means that Bob must've picked them at some point, which means that S would have decreased, a contradiction. Thus the top row is irrelevant and we can remove it from consideration to get a 2020×2021 grid. By the same logic, we can inductively continue to remove rows from the grid until there are no more cells left to operate on, a contradiction, as desired.

6. The number 0 is written on a blackboard. Every minute, Aerith simultaneously replaces every 0 with a 1 and every 1 with a 10. For example, if the current number were 1100, in one minute it would be 101011. She eventually gets tired and leaves, leaving some number N written on the board. If $9|N$, show that $99|N$.

   SOLUTION. Let a_n be the number after n minutes.

After 2 minutes, we note that $a_2 = 10$ is $a_1 = 1$ followed by $a_0 = 0$, so waiting n more minutes gives that a_{n+2} is a_{n+1} followed by a_n. Thus, a_n has F_{n+1} digits, so

$$a_{n+2} = a_{n+1} \cdot 10^{F_{n+1}} + a_n.$$

We now note that, mod 99, $10^{F_{n+1}}$ is equivalent to 1 if $n+1$ is a multiple of 3 and is otherwise equal to 10.

We now use this recursion to write out the repeating period of this sequence mod 99: 0, 1, 10, 2, 12, 23, 44, 67, 21, 79, 1, 89, 0, 89, 98, 79, 78, 67, 55, 23, 87, 2, 89, 1. No non-zero multiple of 9 exists in this sequence, as desired.

7. A certain rectangle can be tiled with a combination of vertical $b \times 1$ tiles and horizontal $1 \times a$ tiles. Show that the rectangle can be tiled with just one of the two types of tiles.

SOLUTION. Let ω be a primitive ab-th root of unity. Fill the cells with complex numbers so that the (i, j) cell (where the top left square is $(1, 1)$) has entry $\omega^{a(i-1)+b(j-1)}$. Then the sum of the entries in a horizontal $1 \times a$ rectangle starting from (i, j) is

$$\omega^{a(i-1)+b(j-1)}(1 + \omega^b + \omega^{2b} + \cdots + \omega^{(a-1)b}) = 0,$$

since ω^b is a primitive a-th root of unity. Similarly, the sum of the entries in a vertical $b \times 1$ rectangle starting from (i, j) is

$$\omega^{a(i-1)+b(j-1)}(1 + \omega^a + \omega^{2a} + \cdots + \omega^{(b-1)a}) = 0,$$

since ω^a is a primitive b-th root of unity. Since the sum of the entries in every tile is zero, the sum of the whole grid must be 0. Suppose the rectangle is $m \times n$. Then the sum of all the entries in the rectangle is

$$(1 + \omega^a + \omega^{2a} + \cdots + \omega^{(m-1)a})(1 + \omega^b + \omega^{2b} + \cdots + \omega^{(n-1)b}) = \frac{\omega^{ma} - 1}{\omega^a - 1} \cdot \frac{\omega^{nb} - 1}{\omega^b - 1} = 0.$$

The only way this can happen is if either $\omega^{ma} = 1$ or $\omega^{nb} = 1$. If $\omega^{ma} = 1$, then since ω is a primitive ab-th root of unity, we must have $b \mid m$, so the rectangle can be tiled with $b \times 1$ tiles only. Similarly, if $\omega^{nb} = 1$, then $a \mid n$, so the rectangle can be tiled with $1 \times a$ tiles only.