Berkeley Math Circle: Monthly Contest 7 Solutions

1. Aerith thinks $(1.4)^{(1.4)^{(1.4)^{\cdot^{\cdot^{\cdot}}}}}$ is well-defined, but Bob thinks it diverges. Who is right?

   SOLUTION. Because $(1.4)^2 = 1.96 < 2$, the tetrations of 1.4 can never surpass 2, so the expression does not diverge to infinity; Aerith is right.

2. On Semi-Predictable Island, everyone is either a liar (who always lies) a truth-teller (who always tells the truth), or a spy (who could do either). Aerith encounters three people and knows that one is a liar, one a truth-teller, and one a spy. She can ask two yes-or-no questions, and all three of them will answer each question. Can she determine which person is which?

   SOLUTION. First she asks the three people, "Are you a spy?" The truth-teller will say "no," the liar will say "yes," and the spy could say either. Either way, two people will give the same answer and the third will give a different number. She can now tell the identity of that third person: if they said "yes" and the other two said "no," they must be the truth-teller, and if they said "no" and the other two said "yes," they must be the liar.

   Now she picks one of the people she does not yet know the identity of (call this person A), and ask the three people whether that person is a spy. The answer of the person whose identity she knows (call this person B) will then tell her whether or not A is a spy. If B is a truth-teller, then A is a spy if and only if B says "yes," and if B is a liar, then A is a spy if and only if B says "no." Either way, she now knows what B is, and that tells her what the third person must be.

3. Find all solutions to $m^4 = n^3 + 137$ over the positive integers.

   SOLUTION. The fourth powers mod 13 are $0, 1, 3, 9$ and the cubes mod 13 are $0, 1, 5, 8, 12$. Therefore, $m^4 - n^3 \equiv 7 \pmod{13}$ is impossible, meaning that there are no solutions.

4. Aerith repeatedly flips a fair coin.

   a) Find the expected number of flips to get two heads in a row.

   b) Find the expected number of flips to get heads followed by tails.

   SOLUTION.

   a) Let x be the expected number of flips. There is a $\frac{1}{4}$ chance that she gets two heads right away. There is a $\frac{1}{2}$ chance the first flip is tails, in which case she is basically starting over after the first flip, so it will take an expected $x + 1$ flips total. Finally, there is a $\frac{1}{4}$ chance the first flip is heads and the second tails, in which case she is basically starting over after 2 flips, so it will take an expected $x+2$ flips total. Thus we get

$$x = \frac{1}{4}\cdot 2 + \frac{1}{2}(x+1) + \frac{1}{4}(x+2) = \frac{3}{2} + \frac{3}{4}x.$$

Solving this gives $x = \boxed{6}$.

b) Let the expected number of flips be y. There is a $\frac{1}{2}$ chance you get tails on the first flip, in which case she is basically starting over, so it will take an expected $y+1$ flips total. If she gets heads on the first flip, then she is just waiting until she first gets tails. She has a $\frac{1}{2}$ chance of getting tails on each flip, so it will be an expected 2 flips before she first gets tails, for an expected 3 flips total. Thus we get

$$y = \frac{1}{2}(y+1) + \frac{1}{2}\cdot 3 = \frac{1}{2}y + 2.$$

Solving this gives $y = \boxed{4}$.

5. Let $ABCDEF$ be a hexagon and let M, N, P, Q, R, S be the midpoints of AB, BC, CD, DE, EF, and FA, respectively. Show that $MQ \perp PS$ if and only if $RN^2 = MQ^2 + PS^2$.

SOLUTION. Let $A = (2a_1, 2a_2)$, $B = (2b_1, 2b_2)$, and so on. Then $M = (a_1 + b_1, a_2 + b_2)$ and $Q = (d_1 + e_1, d_2 + e_2)$, so the slope of MQ is

$$\frac{(a_2+b_2)-(d_2+e_2)}{(a_1+b_1)-(d_1+e_1)}.$$

Similarly, the slope of PS is

$$\frac{(c_2+d_2)-(f_2+a_2)}{(c_1+d_1)-(f_1+a_1)},$$

so the condition that $MQ \perp PS$ is equivalent to

$$\frac{(a_2+b_2)-(d_2+e_2)}{(a_1+b_1)-(d_1+e_1)} \cdot \frac{(c_2+d_2)-(f_2+a_2)}{(c_1+d_1)-(f_1+a_1)} = -1.$$

We have $R = (e_1 + f_1, e_2 + f_2)$ and $N = (b_1 + c_1, b_2 + c_2)$, so

$$RN^2 = [(e_1+f_1)-(b_1+c_1)^2] + [(e_2+f_2)-(b_2+c_2)^2],$$

and similarly,

$$\begin{aligned} MQ^2 &= [(a_1+b_1)-(d_1+e_1)]^2 + [(a_2+b_2)-(d_2+e_2)]^2, \\ PS^2 &= [(c_1+d_1)-(f_1+a_1)]^2 + [(c_2+d_2)-(f_2+a_2)]^2. \end{aligned}$$

If we make the change of variables

$$\begin{aligned} x_1 &= (a_1+b_1)-(d_1+e_1), \\ x_2 &= (a_2+b_2)-(d_2+e_2), \\ y_1 &= (c_1+d_1)-(f_1+a_1), \\ y_2 &= (c_2+d_2)-(f_2+a_2), \end{aligned}$$

The the condition $MQ \perp PS$ becomes

$$\frac{x_2}{x_1} \cdot \frac{y_2}{y_1} = -1 \iff x_2 y_2 = -x_1 y_1,$$

and the condition $RN^2 = MQ^2 + PS^2$ becomes

$$(x_1 + y_1)^2 + (x_2 + y_2)^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 \iff 2x_1 y_1 + 2x_2 y_2 = 0.$$

These two conditions are equivalent, which is what we wanted to show.

6. Aerith picks two numbers $x < y$, and picks one of them to tell to Bob uniformly at random. Is it possible for Bob to have a better than half chance of guessing whether the one he was told is x?

SOLUTION. As an initial attempt, one could choose a threshold T and guess that the number Aerith says is x if and only if it's less than T. This doesn't improve Bob's chances if $x < y < T$ or if $T \leq x < y$, but if $x < T \leq y$, this strategy always works. The trick is to pick T so that one guarantees, no matter what Aerith's strategy is, a nonzero chance of $T \in (x, y]$ for any interval $(x, y]$. One way to achieve this is to pick a random real number $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and set $T = \tan(\theta)$.

7. Let n be a nonnegative integer and let r be an odd number. Show that there is some $0 \leq i < 2^n$ such that

$$\binom{2^n + i}{i} \equiv r \pmod{2^{n+1}}.$$

SOLUTION. We can write $\binom{2^n+i}{i}$ as $\prod_{k=1}^{i} \frac{2^n+k}{k}$. For any $k \leq i < 2^n$, the number of times 2 divides $2^n + k$ is just the number of times 2 divides k, so this product must have an equal number of factors of 2 in the numerator and denominator, and therefore must be odd. Thus, as there are 2^n values of i and 2^n possible values of r (mod 2^{n+1}), the problem is equivalent to showing that $\binom{2^n+i}{i}$ is injective mod 2^{n+1} for $0 \leq i < 2^n$.

Let $0 \leq i < j < 2^n$. Taking the ratio of the corresponding coefficients gives $\prod_{k=i}^{j} \frac{2^n+k}{k}$. Let ν be maximal so that there is a multiple of 2^ν in the range $(i, j]$, and let this multiple be $m2^\nu$ where m is odd.

Now take the ratio mod $2^{n-\nu+1}$. For $k = m2^\nu$, $\frac{2^n+k}{k} = \frac{2^{n-\nu}+m}{m}$, which is not equivalent to one, whereas for $k \neq m2^\nu$, $\frac{2^n+k}{k}$ is equivalent to one. Therefore, the total product is not one mod $2^{n-\nu+1}$, so it is not one mod 2^{n+1}, as desired.