Berkeley Math Circle: Monthly Contest 6 Solutions

1. On Predictable Island, everyone is either a liar (who always lies) or a truth-teller (who always tells the truth). You encounter Aerith who says, "Bob and I are both liars." What are they actually?

   SOLUTION. If Aerith was a truth-teller, she would be lying, because it wouldn't be the case that they're both liars. Thus, she must be a liar. Therefore, the statement must be false, which means Bob has to be a truth-teller.

2. Bob has five airplane tickets with prices \$100, \$120, \$140, \$160, and \$180. Bob gives an offer to Aerith: she can distribute his tickets among two bags, after which, without looking inside, Bob will randomly choose a bag and a ticket from it for Aerith to keep. What strategy should Aerith use to maximize the expected value of her ticket?

   SOLUTION. Let the first bag be the one with less tickets. Let $a < b$ and S, T be the numbers and prices of tickets in the respective bags. We have $a + b = 5$ and $S + T = \$700$. The expected values of random tickets for each bag is then S/a and T/b, the average of which is

   $$\frac{S/a + T/b}{2} = \frac{bS + aT}{2ab} = \frac{(b-a)S + a \cdot \$700}{2ab}.$$

   For fixed $a < b$, this is maximized when S is a large as possible. Thus if $(a, b) = (1, 4)$, the first bag then should contain \$180, whereas for $(a, b) = (2, 3)$, it should contain \$180+\$160=\$340.

   Using the first strategy, she has an expected value of $\frac{\$180/1+\$(700-180)/4}{2} = \$155$, and using the second, she has an expected value of $\frac{\$340/2+\$(700-340)/3}{2} = \$145$, so the best strategy is the first strategy: put the \$180 ticket in one bag and the rest in another.

3. In $\triangle ABC$, let O be the circumcenter, D the foot of the altitude from A to BC, and E the foot of the altitude from B to AC. Show that $DE \perp CO$.

   SOLUTION. By AA, $\triangle ACD \sim \triangle BCE$, as $\angle C$ is shared and both are right triangles. Thus $AC/CD = BC/CE$. It follows by SAS that

   $$\triangle ABC \sim \triangle DEC \implies \angle CDE = \angle BAC.$$

   Since O is the circumcenter of $\triangle ABC$, $\angle BOC = 2\angle BAC$, and since $\triangle BOC$ is isosceles,

   $$\angle OCB = \angle OBC = 90^\circ - \angle BAC.$$

   Thus, $\angle CDE + \angle OCB = 90^\circ$, which implies that $DE \perp OC$.

4. Find the remainder when $10^{2^0} + 10^{2^1} + \cdots + 10^{2^{2021}}$ is divided by $\underbrace{44\cdots44}_{44}$.

SOLUTION. Let $N = 10^{2^0} + 10^{2^1} + \cdots + 10^{2^{2021}}$. By CRT, it suffices to find N mod 4 and mod $M = \underbrace{11\cdots11}_{44}$.

Modulo 4, the remainder is 2. Modulo $M = (10^{44}-1)/9$, $10^{44} = 1$, so $10^{2^i} \pmod{M}$ is determined by $2^i \pmod{44}$.

Mod 44, 2^0 is 1 and 2^1 is 2. The remaining powers of two are all divisible by 4. Furthermore, the period of $2 \pmod{11}$ is 10, so the remaining 2020 powers of 2 are in fact divided equally among 10 nonzero remainders that are multiples of 4. Therefore, $N \pmod{M}$ is

$$10^1 + 10^2 + \frac{2020}{10} \cdot \sum_{i=1}^{10} 10^{4i} = 20202020202020202020202020202020202020202020110,$$

which is $2 \pmod{4}$, and is therefore the answer.

5. Aerith has an unlimited number of coins of values $5, m-1$, and m, and 71 is the greatest value that cannot be made with a combination of these coins. Find all possible values of m.

SOLUTION. By the Chicken McNugget Theorem, the largest coin value that can't be achieved with 5 and $m-1$ is

$$5(m-1) - 5 - (m-1) = 4m - 9 \geq 71,$$

so $4m \geq 80$ and thus $m \geq 20$.

Since 71 is not possible, there cannot be a number less than 71 which is $1 \pmod{5}$ and divisible by m or $m-1$, since then you could add some number of 5's to it to get 71. In particular, m and $m-1$ cannot be $1 \pmod{5}$, so m can't be 1 or 2 (mod 5).

Since all numbers above 71 are achievable, 76 has to be achievable, and without using any 5's. If $m \geq 36$, the only way this can happen is if it's either $2m$ or $2(m-1)$, so $m = 38$ or $m = 39$. If $m = 39$, then there's no way to get 72. However, if $m = 38$, then we can get:

$$\begin{aligned}
72 &= 37 + 7 \cdot 5, \\
73 &= 38 + 7 \cdot 5, \\
74 &= 2 \cdot 37, \\
75 &= 37 + 38, \\
76 &= 2 \cdot 38.
\end{aligned}$$

We also can't get 71, so $m = 38$ is a solution.

If $m \leq 37$, then $2m$ also can't be $1 \pmod{5}$, so m can't be $3 \pmod{5}$. Also, $2(m-1)$ can't be $1 \pmod{5}$, so m can't be $4 \pmod{5}$. Thus m has to be a multiple of 5. But then $m-1 \equiv 4 \pmod 5$, so the only way we could get 76 is if it's $4(m-1)$, which gives $m=20$. This works since 71 can't be achieved but

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\begin{aligned}
72 &= 3\cdot 19 + 3\cdot 5,\\
73 &= 2\cdot 19 + 20 + 3\cdot 5,\\
74 &= 19 + 2\cdot 20 + 3\cdot 5,\\
75 &= 15\cdot 5,\\
76 &= 2\cdot 38.
\end{aligned}
$$

Thus, the two solutions are $\boxed{m=20}$ and $\boxed{m=38}$.

6. Find all functions $f:\mathbb{Q}\to\mathbb{R}$ from rational to real numbers such that for all rational p, q,

$$f(p+2q)-f(p-2q)=2(f(p+q)-f(p-q)).$$

SOLUTION. Let $P(p,q)$ be the given condition. Expanding $P(p-q,q)+2P(p,q)+P(p+q,q)$ gives

$$f(p+3q)-f(p-3q)=3(f(p+q)-f(p-q)).$$

Let F be the quadratic that equals f at $-1, 0, 1$. Plugging in $p=0.5, q=0.5$, we get that it also intersects f at 2. Plugging in $p=1.5, q=0.5$ then gives that it also intersects f at 3, and continuing this induction gives that it is equal to f for all positive integers. Continuing this reasoning in the other direction gives that it is equal to f for all negative integers as well.

We now claim that f is in fact equal to F. Let r/s be a rational number. Consider the arithmetic sequence from $-r$ to r with difference r/s. By similar reasoning, f is a quadratic over this sequence. However, F contains three of the terms: $-r, 0, r$. Therefore, f is equal to F over this sequence; in particular, $f(r/s)=F(r/s)$.

Therefore, f is a quadratic, and all quadratics work, so the answer is all quadratics.

7. In the game Pogémon, Pogé Balls contain Pichagus with a 50% chance and otherwise contain nothing. The new Pogémon universe started with one Pichagu and has existed for five hours. Every hour, every Pichagu at the time opened a new Pogé Ball.

a) What is the expected number of Pichagus created in the last hour?

b) Every two currently existing Pichagus will now battle each other. What is the expected number of battles?

SOLUTION. This problem was inspired by `https://xkcd.com/1516/`.

a) Every hour, the expected number of Pichagus increases 50%. Thus, the expected number of Pichagus after 4 hours is $\left(\frac{3}{2}\right)^4$. Each of these Pichagus has a 50% chance of creating a new Pichagu in the last hour, so the answer is $\frac{1}{2}\left(\frac{3}{2}\right)^4 = \frac{81}{32} = 2.53125$.

b) We show via induction that the answer for n hours is $E(n) := \left(\frac{3}{2}\right)^{2n-1} - \left(\frac{3}{2}\right)^{n-1}$. This is clearly true for $n = 0$; no battles would happen at that point.

Now we show that this is true for $n+1$ given that it's true for n. After one hour, there are two cases: either there is one Pichagu or there are two.

In the first case, the expected value is just $E(n)$ as there are n hours left.

Otherwise, say that the Pichagus are divided into two "factions": the Pichagus that descend from the original Pichagu and the ones that descend from the second one. Each faction has an expected value of $E(n)$ internal battles (battles between two members of the faction). Additionally, as in part (a), each faction has an expected value of $\left(\frac{3}{2}\right)^n$ members, so there will be an expected value of $\left(\left(\frac{3}{2}\right)^n\right)^2$ battles between factions. Thus in this case there will be an expected value of $2E(n) + \left(\frac{3}{2}\right)^{2n}$ battles.

Combining the cases we have

$$\begin{aligned} E(n+1) &= \frac{1}{2}E(n) + \frac{1}{2}\left(2E(n) + \left(\frac{3}{2}\right)^{2n}\right) \\ &= \frac{1}{2}\left(\frac{3}{2}\right)^{2n} + \frac{3}{2}E(n) \\ &= \frac{3}{4}\left(\frac{3}{2}\right)^{2n-1} + \frac{3}{2}\left(\left(\frac{3}{2}\right)^{2n-1} - \left(\frac{3}{2}\right)^{n-1}\right) \\ &= \left(\frac{3}{2}\right)^{2n+1} - \left(\frac{3}{2}\right)^{n}, \end{aligned}$$

as desired. Plugging in $n = 5$ gives $\left(\frac{3}{2}\right)^9 - \left(\frac{3}{2}\right)^4 = \frac{3^9 - 2^5 \cdot 3^4}{2^9} = \frac{17091}{512} = 33.380859375$.