Berkeley Math Circle: Monthly Contest 5 Solutions

1. In a trapezoid, the midsegment has length 17 and the distance between the midpoints of the diagonals is 7. Find the lengths of the bases.

   SOLUTION. Let a and b be the bases, with $a > b$. The length of the midsegment is the average of the bases, so $(a+b)/2 = 17$, and the distance between the midpoints of the diagonals is half their difference, so $(a-b)/2 = 7$. Adding the two equations gives $a = \boxed{24}$, and subtracting gives $b = \boxed{10}$.

2. Aerith has 5 coins, all with heads facing up. She wants to flip them so that they all have tails facing up.

   a) If she must flip exactly three coins at a time (from heads to tails or vice versa), is this possible?

   b) What if she must flip exactly two coins at a time?

   For each part, either show a way it can be done or prove that it is impossible.

   SOLUTION.

   a) This is possible. One solution is $HHHHH \to TTTHH \to THHTH \to TTTTT$.

   b) This is not possible. Since you are always flipping two coins at a time, the number of heads will always go down by two, go up by two, or stay the same. Since the number of heads starts out odd, it must always stay odd, and thus can never be zero.

3. Show that $a^{1729} \equiv a \pmod{1729}$ for all positive integers a.

   SOLUTION. Since $1729 = 7 \cdot 13 \cdot 19$, by the Chinese Remainder Theorem it suffices to show that $a^{1729} \equiv a$ in mod 7, mod 13, and mod 19. This follows from Fermat's Little Theorem, since

$$a^{1729} \equiv (a^6)^{288} \cdot a \equiv a \pmod{7},$$
$$a^{1729} \equiv (a^{12})^{144} \cdot a \equiv a \pmod{13},$$
$$a^{1729} \equiv (a^{18})^{96} \cdot a \equiv a \pmod{17}.$$

4. How many solutions does $26 = twelve + eleven + two + one$ have over the positive integers? (Each letter is a variable, and letters in the same word are multiplied.)

   SOLUTION. Factoring, we get $26 = twelve + eleven + two + one = (elve + o)(tw + ne)$.

Both factors are at least $1+1=2$, so there are two cases: either $elve+o=2$ and $tw+ne=13$ or $elve+o=13$ and $tw+en=2$.

In the first case, $elve=o=1$, so $e=l=v=o=1$. We are then left with $tw+n=13$.

In the second case, $tw=en=1$. so $t=w=e=n=1$. We are then left with $lv+o=13$.

Each case has the same number of solutions as $ab+c=13$. For each a, there are $\left\lfloor \frac{12}{a} \right\rfloor$ multiples of a under 13. Thus the number of solutions to this equation is

$$\sum_{a=1}^{12} \left\lfloor \frac{12}{a} \right\rfloor = 12+6+4+3+2+2+1+1+1+1+1+1=35,$$

so the number of solutions to the original problem is $2 \cdot 35 = 70$.

5. Aerith writes 100 positive numbers on a blackboard. Every minute, Bob either replaces one number x with $\frac{1}{x}$, or replaces two numbers x, y with $\frac{xy+1}{x+y}$. Given that after 2021 minutes, there is only one number left, show that this number only depends on Aerith's initial numbers.

SOLUTION. Let $x \star y = \frac{xy+1}{x+y}$. One can check that $x \star \frac{1}{y} = \frac{1}{x \star y}$. Because of the latter property, Bob would get the same number by doing all $\star$ operations first and doing all multiplicative inverses last. Every $\star$ decreases the total number of numbers by 1, so the number of times $\star$ is used is always 99, and the number of inversions is therefore constant. Thus, all one needs to show is that the number after 99 $\star$ operations is only dependent on Aerith's initial numbers, which follows from checking that $\star$ is commutative and associative.

6. Show that the sum $AP^4 + BP^4 + CP^4$ does not depend on P, where P is a point on the circumcircle of equilateral triangle $\triangle ABC$.

SOLUTION. WLOG assume P is between A and B, and let s be the side length of $\triangle ABC$. By Ptolemy's Theorem,

$$(PA+PB) \cdot s = PA \cdot BC + PB \cdot AC = PC \cdot AB = PC \cdot s \implies PA+PB=PC.$$

By the law of cosines on $\triangle PAB$,

$$PA^2 + PB^2 - 2PA \cdot PB \cos \angle PAB = AB^2.$$

Since $\angle PAB = 120^\circ$ and $\cos 120^\circ = -\frac{1}{2}$, this becomes

$$PA^2 + PB^2 + PA \cdot PB = s^2.$$

Squaring both sides gives

$$PA^4 + 2PA^3 \cdot PB + 3PA^2 \cdot PB^2 + 2PA \cdot PB^3 + PB^4 = s^4.$$

Doubling both sides and rearranging gives

$$\begin{aligned} 2s^4 &= 2PA^4 + 4PA^3 \cdot PB + 6PA^2 \cdot PB^2 + 4PA \cdot PB^3 + 2PB^4 \\ &= PA^4 + PB^4 + (PA + PB)^4. \end{aligned}$$

Using the fact that $PA + PB = PC$, we get

$$PA^4 + PB^4 + PC^4 = 2s^4.$$

7. "Very Frustrating Game" has six levels. When a level is attempted, the player goes to the next level if they succeed, but back to the previous level if they fail (or if they are on level 1 they restart).

   a) Aerith has a $\frac{1}{2}$ success rate on all levels. How many level attempts on average would it take her to complete the game?

   b) Bob has a $\frac{1}{3}$ success rate on all levels. How many level attempts on average would it take him to complete the game?

SOLUTION.

a) Let $A(x) = 42 - (x + x^2)$. One can check that $A(x) = 1 + \frac{1}{2}A(x+1) + \frac{1}{2}A(x-1)$, that $A(0) = A(-1)$, and that $A(6) = 0$. The answer when starting on level $n+1$ is therefore $A(n)$. Plugging in $n = 0$ gives 42.

b) Let $B(x) = 360 - 3(2(2^x - 1) - x)$. One can check that $B(x) = 1 + \frac{1}{3}B(x+1) + \frac{2}{3}B(x-1)$, that $B(0) = B(-1)$, and that $B(6) = 0$. The answer when starting on level $n + 1$ is therefore $B(n)$. Plugging in $n = 0$ gives 360.