Berkeley Math Circle: Monthly Contest 4 Solutions

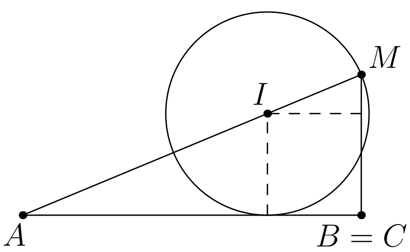
1. Aerith bakes some cookes. On the first day, she gives away 1 cookie and then 1/8 of the remaining cookies; on the second day, she gives away 2 cookies and then 1/8 of the remaining cookies, and so on. On the 7th day, she gives away 7 cookies and then there are none left. How many cookies did she bake?

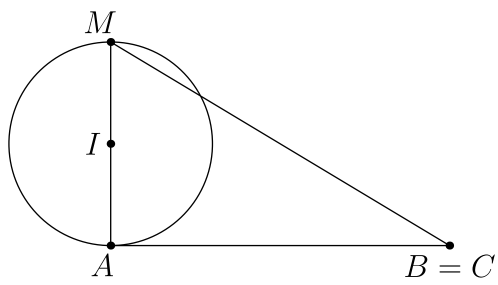
   SOLUTION. The number of cookies is $\boxed{49}$. Working backwards, on the 6th day, 1/8 of the remaining cookies must have been 1 cookie (since there were 7 left after that), so there were $6 + 1 + 7 = 14$ cookies at the start of the day. Similarly, since the 5th day there were 14 cookies left, the 1/7 of the remaining cookies given away was 2 cookies, so there were $5 + 2 + 14 = 21$ cookies to start with.

   Continuing, at the start of the 4th day, there were $4+3+21 = 28$, on the third day, there were $3 + 4 + 28 = 35$, on the second day there were $2 + 5 + 25 = 42$, and on the first day, there were $1 + 6 + 42 = 49$ cookies to start.

2. Points A, B, C, M are such that B, M, C lie on a line, $AB = AC = 353$, and $BM = MC$. Point I is on segment AM such that the circle centered at I through M has radius 106 and is tangent to AB and AC. Given that there are two possible areas of right triangle AMB, find the larger one.

   SOLUTION. The two configurations that work are the configuration when $B = C$ and angle A is right and the configuration when $B = C$ and angle B is right. In the second case, the height from M is $2 \cdot 106$, which is larger than in the other case, The answer is thus $106 \cdot 353 = 37418$.





3. Let A and B be diagonally opposite vertices of a cube. An ant is crawling on a cube starting from A, and each second it moves at random to one of the three vertices adjacent to its current one. Find the expected number of steps for the ant to get to vertex B.

   SOLUTION. Let C_1, C_2, C_3 be the vertices of the cube adjacent to A, and D_1, D_2, D_3 the vertices adjacent to B. Let x be expected time to get to B starting from A. From A, you have to go to C_1, C_2, or C_3, so the expected time to get to B from one of these vertices is $x - 1$.

   Let y be the expected number of steps to get to B from D_1, D_2, or D_3 (these are all the same by symmetry). From these points, you have a $\frac{1}{3}$ chance of going to B

immediately, and a $\frac{2}{3}$ chance of going to one of C_1, C_2, or C_3, from which it will take an expected $x-1$ additional steps to get to B (plus the one you already took). Thus, the expected number of steps to get from D_1, D_2, or D_3 to B is

$$y = \frac{1}{3} \cdot 1 + \frac{2}{3}x.$$

Now, from C_1, C_2, or C_3, you have a $\frac{1}{3}$ chance of going to A, in which case the expected number of additional steps to get to B will be x, and you have a $\frac{2}{3}$ chance of going to one of D_1, D_2, or D_2, in which case the expected number of additional steps will be y. This gives

$$x - 1 = \frac{1}{3}(1+x) + \frac{2}{3}(1+y) = 1 + \frac{1}{3}x + \frac{2}{3}y.$$

Substituting in for y using the first equation gives

$$x = 2 + \frac{1}{3}x + \frac{2}{3}\left(\frac{1}{3} + \frac{2}{3}x\right) = \frac{20}{9} + \frac{7}{9}x.$$

Solving this gives $\boxed{x = 10}$.

4. Let $S \subset \mathbb{N}$ be a set of positive integers whose product is 2021 times its sum.
   a) Given that S has five (or more) elements, what its minimum possible sum?
   b) Given that S has exactly four elements, what its minimum possible sum?
   c) Given that S has exactly three elements, what its minimum possible sum?

SOLUTION. 2021 is prime factorized as $43 \cdot 47$. There must therefore be at least one multiple of 43 and at least one multiple of 47.

a) The answer is $105 = 47 + 43 + 7 + 5 + 3$. $2 \cdot 43 + 47$ is already too much, so any smaller answer would have to involve both 43 and 47. If T is $S \setminus \{43, 47\}$, the product of the elements of T is thus $43 + 47 = 90$ more than the sum of elements of T. However, one can check that this cannot hold for any product under 105.

b) The answer is $112 = 47 + 43 + 14 + 8$. We define T similarly and let its elements be a and b. We have $a + b + 90 = ab$, which can be rewritten as $(a-1)(b-1) = 91 = 7 \cdot 13$. The minimum value of $a + b - 2 = (a-1) + (b-1)$ is thus $7 + 13 = 20$, giving an answer of $22 + 90$, as desired.

c) The answer is $240 = 94 + 86 + 60$. Let the numbers be $47a, 43b, x$, so that $47a + 43b + x = abx$. Solving for x gives $x = 47a + 43b/(ab-1)$. In order to get less than 240, we must have $a + b < 240/43 < 6$. Checking all combinations gives the answer.

5. Determine all complex numbers w such that

$$10|w|^2 = 2|w+2|^2 + |w^2+1|^2 + 20.$$

SOLUTION. Let $w = a + bi$, where a and b are real. The equation becomes

$$\begin{aligned}10(a^2+b^2) &= 2[(a+2)^2+b^2] + (a^2-b^2+1)^2 + (2ab)^2 + 20\\ &= 2a^2+8a+8+2b^2+(a^4-2a^2b^2+b^4)+2(a^2-b^2)+1+4a^2b^2+20\\ &= 4a^2+8a+(a^2+b^2)^2+29.\end{aligned}$$

If we set $r = a^2 + b^2$, this becomes

$$r^2 - 10r + 4a^2 + 8a + 29 = (r-5)^2 + 4(a+1)^2 = 0.$$

The only way we get equality is if $r = 5$ and $a = -1$, which means $b = \pm 2$. Thus the solutions are $\boxed{w = -1 \pm 2i}$.

6. Squares $MATH$, $GREW$, $UNDO$ are such that MUG, RAN, TED are all lines. Show that WHO is a line as well.

SOLUTION. Fix lines a, b, c. The problem is equivalent to showing that for squares $ABCD$ (with vertices ordered clockwise) where $A \in a, B \in b, C \in c$, the locus of D is a line.

There are two cases. Either $a \not\perp c$ or $a \perp c$.

In the first case, write B as a linear function of some parameter t. Let a' be the rotation of a counter-clockwise about B by $90°$. $C = a' \cap c$ is then given by a linear function as well. Likewise, A is given by a linear function. Therefore by vector addition, $D = A + C - B$ is linear, giving the desired.

Otherwise, WLOG let a be the line $x = y$ and c be the line $x + y = 0$. Let $A = (p, p)$ and $C = (-q, q)$. We then have $D = (0, p + q)$, so D lies on the y-axis, giving the desired.

7. In a country with n cities, there is a recurrent one-way flight between every pair of cities. Every flight has a constant price in the range \$100, \$120, \$140, \$160, \$180. A \$$N$ flight ticket gives unlimited access to flights which cost \$$N$, and tickets can be traded for tickets of lower prices. For example, with a \$160 ticket, Bob could take a \$160 flight, trade his ticket for a \$120 ticket, then take a \$120 flight.

Aerith loves flying and wonders how many successive flights she can take with one ticket. What is the minimum n needed to guarantee that she can take 4 such flights in a row?

SOLUTION. For each city C, let $f_C(\$N)$ be the maximum distance one can travel starting at C with an \$$N$ ticket. If Aerith cannot take 4 flights in a row, f_C is an non-decreasing function that remains between 0 and 3, of which there are $\left(\!\binom{4}{5}\!\right) = \binom{8}{5} = 56$. No two cities C_1 and C_2 can have the same f, because if there is a flight of price \$$N$ from C_1 to C_2, $f_{C_1}(\$N) \geq f_{C_2}(\$N) + 1$. 57 thus forces there to be a chain of 4 flights in a row.

To show that it is possible for all chains of flights to have length at most 3 for 56 cities, we reverse this proof — correspond each city C to one non-decreasing function

F from tickets to $\{0,1,2,3\}$, and for every two cities where N is maximal so that $f_{C_1}(\$N) \neq f_{C_2}(\$N)$, let there be a flight from C_1 to C_2 with price $\$N$. One can see that $f_C = F_C$, giving the desired.