1. Each of Alice, Bob, and Carol is either a consistent truth-teller or a consistent liar.

Alice states: “At least one of Bob or Carol is a truth-teller.” Bob states: “Alice and Carol are both truth-tellers.” Carol states: “If Alice is a truth-teller, so too is Bob.” Must they all be truth-tellers?

**SOLUTION.** Yes. If Carol were a liar, Alice would have to be a truth-teller while Bob would have to be a liar. However, Bob would then be telling the truth, a contradiction.

Thus Carol is telling the truth. Alice’s statement is then true as well, and thus Bob’s statement is also true. Hence, all logicians must be telling the truth.

2. Let $P$ be a polynomial with integer coefficients. Let $S$ be the set of integers $n$ for which $P(n)/n$ is an integer. Show that $S$ contains either finitely many integers, or all but finitely many integers.

**SOLUTION.** Let $c$ be the constant coefficient of $P$, so that $P(x)$ is of the form $Q(x)x + c$. If $n|P(n)$, we then have $n|(Q(n)n + c)$, or $n|c$. $S$ is thus the set of divisors of $c$. If it is not finite, $c$ must then be 0, and $S$ is the set of all nonzero integers, as desired.

3. If $a, b,$ and $c$ are positive real numbers with $2a + 4b + 8c = 16$, what is the largest possible value of $abc$?

**SOLUTION.** By AM-GM,

$$\frac{16}{3} = \frac{2a + 4b + 8c}{3} \geq \sqrt[3]{(2a) \cdot (4b) \cdot (8c)} = 4\sqrt[3]{abc}.$$ 

Rearranging, we get

$$\sqrt[3]{abc} \leq \frac{4}{3} \iff abc \leq \frac{64}{27}.$$ 

This maximum is indeed attainable if we set $2a = 4b = 8c$, since that is the equality case of AM-GM, which means $2a = 4b = 8c = \frac{16}{3}$, so $a = \frac{8}{3}, b = \frac{4}{3}, c = \frac{2}{3}$. We conclude that the maximum is indeed $\frac{64}{27}$.

4. Let $P$ be a 2023-sided polygon. All but one side has length 1. What is the maximum possible area of $P$?

**SOLUTION.** First, we claim $P$ must be convex to maximize its area. If not, let $A$ and $B$ be consecutive vertices on the perimeter of its convex hull that aren’t
consecutive vertices of $P$. Reflecting the path between $A$ and $B$ over line $AB$ must increase the area of $P$ as the new shape strictly contains $P$.

Thus we assume $P$ is convex. Let $\ell$ be the line containing the side with length not equal to 1. Let $P'$ be the reflection of $P$ over $\ell$. By convexity, $P$ and $P'$ do not overlap, so the union of $P$ and $P'$ is a polygon, specifically an equilateral 4044-gon with sides of length 1.

The area of an equilateral polygon is maximized when it is regular, so this union has maximum area that of a regular 4044-gon, which is $1011 \cot \pi/4044$. Thus, the answer is half of this, i.e.

$$\frac{1011}{2} \cdot \cot \frac{\pi}{4044}.$$ 

5. Suppose you have only an unmarked straightedge (no compass), and you are given a line segment $AB$ with midpoint $O$ and a point $P$ not on line $AB$.

(a) Construct a line through $P$ parallel to $AB$.

(b) If you are also given the circle with center $O$ and radius $OA$ and $P$ does not lie on the circle, construct a line through $P$ perpendicular to $AB$.

**SOLUTION.**

(a) Extend line $AP$ to some point $Q$ on the opposite side of $P$ from $A$. Let $R$ be the intersection of lines $QO$ and $PB$, and let $S$ be the intersection of lines $AR$ and $QB$, as shown below. We claim that $PS$ is parallel to $AB$.

![Diagram](image)

By Ceva’s theorem,

$$\frac{QP}{PA} \cdot \frac{AO}{OB} \cdot \frac{BS}{SQ} = 1.$$ 

It follows that

$$\frac{QP}{PA} = \frac{QS}{QB},$$

which means $\triangle QPS \sim \triangle QAB$, therefore $\angle QPS = \angle QAB$ and so $PS \parallel AB$.

(b) Let $PA$ and $PB$ meet the circle again at points $Q$ and $R$, respectively, and let $S$ be the intersection of $AR$ and $BQ$. 

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Then \( \angle AQB = \angle ARB = 90^\circ \) since both angles are inscribed in a semicircle, so 

\( S \) is the orthocenter of \( \triangle ABP \), which means \( PS \) is the desired perpendicular.

6. Let \( A \) be a set of size 2023. Find the maximum number of pairs of elements \( x, y \in A \) so that \( x - y \) is a power of \( e \).

**SOLUTION.**

Let \( a_n \) be the maximum possible number of such pairs for a set of size \( n \). Let \( s_2(n) \) be the number of ones in \( n \)’s binary representation. Let \( S(n) = \sum_{k=0}^{n-1} s_2(k) \). We show that \( a_n = S(n) \).

For the construction, we can take the binary representations of all numbers from 0 to \( n - 1 \), and interpret them as numbers “base \( e \).” Every \( x \) corresponding to some integer \( 0 \leq k < n \) then has \( s_2(k) \) working values of \( y \), corresponding to all ways to replace a 1 with a 0 in \( k \)’s binary representation.

For optimality, we use strong induction. The base case of \( n = 1 \) holds as \( a_1 = 0 = s_2(0) \).

Now assume \( n > 1 \). If \( A \) has no working pairs \( x, y \), we are done. Otherwise, let \( t \) be an integer so that there is at least one pair \( x, y \in A \) so that \( x - y = e^t \).

Let \( G \) be the graph of such pairs in \( A \). If \( G \) is not connected, we can increase the number of edges of \( G \) by shifting the vertices of one component of \( G \) to create at least one edge to another component. Thus we can assume that all elements of \( A \) are sums of powers of \( e \). Let \( F \) for an element \( z \in A \), let \( c_z \) be the coefficient of \( e^t \) in the representation of \( A \) as a sum.

Let \( X \) be the set of \( z \) so that \( c_z \geq c_x \) and let \( Y \) be the set of \( z \) so that \( c_z \leq c_y \). Note that \( X \sqcup Y = A \). By strong induction, there are at most \( a_{|X|} \) working pairs in \( X \), and at most \( a_{|Y|} \) pairs in \( Y \). By definition of \( X \) and \( Y \), any pair between them can only have one possible difference, namely \( e^t \). Thus, there are at most \( \min(|X|, |Y|) \) pairs between them.

Thus, we have the recurrence \( a_n \leq \max_{x+y=n} (a_X + a_Y) + \min(X,Y) \). It thus suffices to show that if \( Y \geq X \), \( S(X + Y) - S(Y) \geq S(X) + X \), which expands to
\[ \sum_{k=0}^{X-1} s_2(k) \geq \sum_{k=0}^{X-1} (1 + s_2(k)) \]. An exercise to the interested reader is to show this by strong induction on \( X \).

Now it remains to evaluate \( S(2023) \). By linearity of expectation, \( S(2048) \) is equal to \( 2048 \cdot 11/2 \). For every number from \( 2032 = 2048 - 16 \) to \( 2048 \), 7 digits must be 1 and the remaining four each have a half chance of being 1, giving \( S(2048) - S(2032) = 16 \cdot (7 + 4/2) \). Similarly \( S(2032) - S(2024) = 8 \cdot (7 + 3/2) \), \( S(2024) - S(2023) \) is just the number of ones in \( 2023 = 11111100111_2 \) is 9. Thus the answer is

\[ 2048 \cdot 11/2 - 16 \cdot (7 + 4/2) - 8 \cdot (7 + 3/2) - 9 = 11043. \]

7. Show that for sufficiently large primes \( p \), there is an Eulerian circuit on the complete graph with \( p \) vertices that does not contain any cycles of length at most 2023.

**SOLUTION.** Take a generator \( g \) (mod \( p \)) so that \( g \) is not equivalent to anything the form \(-a/b\) for integers \( a, b \leq 2023 \). For big enough primes, such a \( g \) must exist as there are at most a constant number of such fractions.

Now number the vertices with the residues mod \( p \). For any residue \( r \) (mod \( p \)), consider the sequence of vertices \( P_r \) formed by the multiples of \( r \), starting from \( r \) and ending at \( pr \equiv 0 \) (mod \( p \)). We claim that the concatenation of \( P_1, P_g, P_{g^2}, \ldots \) works.

Clearly no \( P_r \) contains a cycle. Thus, if a cycle were to exist, it would have to be formed by the end of one \( P \) and the start of another. In other words, we would have \( cg^{i+1} \equiv (p-d)g^i \) (mod \( p \)) for \( c + d \leq 2023 \). However, this is impossible by choice of \( g \).