Berkeley Math Circle: Monthly Contest 2 Solutions

1. Each of Alice, Bob, and Carol is either a consistent truth-teller or a consistent liar.

   Alice states: "At least one of Bob or Carol is a truth-teller." Bob states: "Alice and Carol are both truth-tellers." Carol states: "If Alice is a truth-teller, so too is Bob." Must they all be truth-tellers?

   SOLUTION. Yes. If Carol were a liar, Alice would have to be a truth-teller while Bob would have to be a liar. However, Bob would then be telling the truth, a contradiction.

   Thus Carol is telling the truth. Alice's statement is then true as well, and thus Bob's statement is also true. Hence, all logicians must be telling the truth.

2. Let P be a polynomial with integer coefficients. Let S be the set of integers n for which $P(n)/n$ is an integer. Show that S contains either finitely many integers, or all but finitely many integers.

   SOLUTION. Let c be the constant coefficient of P, so that $P(x)$ is of the form $Q(x)x+c$. If $n|P(n)$, we then have $n|(Q(n)n+c)$, or $n|c$. S is thus the set of divisors of c. If it is not finite, c must then be 0, and S is the set of all nonzero integers, as desired.

3. If a, b, and c are positive real numbers with $2a + 4b + 8c = 16$, what is the largest possible value of abc?

   SOLUTION. By AM-GM,

$$\frac{16}{3} = \frac{2a+4b+8c}{3} \geq \sqrt[3]{(2a)\cdot(4b)\cdot(8c)} = 4\sqrt[3]{abc}.$$

   Rearranging, we get

$$\sqrt[3]{abc} \leq \frac{4}{3} \iff abc \leq \frac{64}{27}.$$

   This maximum is indeed attainable if we set $2a = 4b = 8c$, since that is the equality case of AM-GM, which means $2a = 4b = 8c = \frac{16}{3}$, so $a = \frac{8}{3}, b = \frac{4}{3}, c = \frac{2}{3}$. We conclude that the maximum is indeed $\boxed{\frac{64}{27}}$.

4. Let P be a 2023-sided polygon. All but one side has length 1. What is the maximum possible area of P?

   SOLUTION. First, we claim P must be convex to maximize its area. If not, let A and B be consecutive vertices on the perimeter of its convex hull that aren't consecutive vertices of P. Reflecting the path between A and B over line AB must increase the area of P as the new shape strictly contains P.

Thus we assume P is convex. Let ℓ be the line containing the side with length not equal to 1. Let P' be the reflection of P over ℓ. By convexity, P and P' do not overlap, so the union of P and P' is a polygon, specifically an equilateral 4044-gon with sides of length 1.

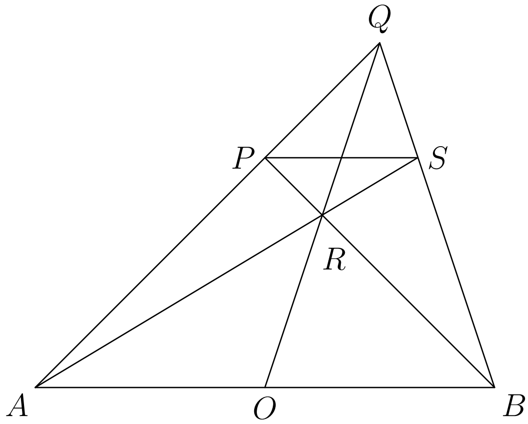
The area of an equilateral polygon is maximized when it is regular, so this union has maximum area that of a regular 4044-gon, which is $1011 \cot \pi/4044$. Thus, the answer is half of this, i.e.

$$\frac{1011}{2} \cdot \cot \frac{\pi}{4044}.$$

5. Suppose you have only an unmarked straightedge (no compass), and you are given a line segment AB with midpoint O and a point P not on line AB.
   (a) Construct a line through P parallel to AB.
   (b) If you are also given the circle with center O and radius OA and P does not lie on the circle, construct a line through P perpendicular to AB.

SOLUTION.

(a) Extend line AP to some point Q on the opposite side of P from A. Let R be the intersection of lines QO and PB, and let S be the intersection of lines AR and QB, as shown below. We claim that PS is parallel to AB.



By Ceva's theorem,

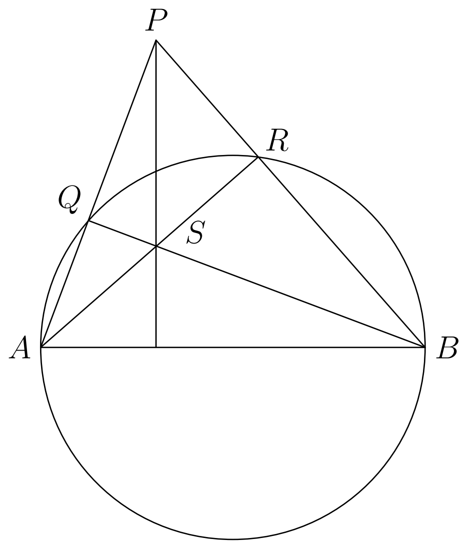
$$\frac{QP}{PA} \cdot \frac{AO}{OB} \cdot \frac{BS}{SQ} = 1.$$

It follows that

$$\frac{QP}{PA} = \frac{QS}{QB},$$

which means $\triangle QPS \sim \triangle QAB$, therefore $\angle QPS = \angle QAB$ and so $PS \parallel AB$.

(b) Let PA and PB meet the circle again at points Q and R, respectively, and let S be the intersection of AR and BQ.



Then $\angle AQB = \angle ARB = 90^\circ$ since both angles are inscribed in a semicircle, so S is the orthocenter of $\triangle ABP$, which means PS is the desired perpendicular.

6. Let A be a set of size 2023. Find the maximum number of pairs of elements $x, y \in A$ so that $x - y$ is a power of e.

SOLUTION.

Let a_n be the maximum possible number of such pairs for a set of size n. Let $s_2(n)$ be the number of ones in n's binary representation. Let $S(n) = \sum_{k=0}^{n-1} s_2(k)$. We show that $a_n = S(n)$.

For the construction, we can take the binary representations of all numbers from 0 to $n-1$, and interpret them as numbers "base e". Every x corresponding to some integer $0 \le k < n$ then has $s_2(k)$ working values of y, corresponding to all ways to replace a 1 with a 0 in k's binary representation.

For optimality, we use strong induction. The base case of $n = 1$ holds as $a_1 = 0 = s_2(0)$.

Now assume $n > 1$. If A has no working pairs x, y, we are done. Otherwise, let t be an integer so that there is at least one pair $x, y \in A$ so that $x - y = e^t$.

Let G be the graph of such pairs in A. If G is not connected, we can increase the number of edges of G by shifting the vertices of one component of G to create at least one edge to another component. Thus we can assume that all elements of A are sums of powers of e. Let For an element $z \in A$, let c_z be the coefficient of e^t in the representation of A as a sum.

Let X be the set of z so that $c_z \ge c_x$ and let Y be the set of z so that $c_z \le c_y$. Note that $X \sqcup Y = A$. By strong induction, there are at most $a_{|X|}$ working pairs in X, and at most $a_{|Y|}$ pairs in Y. By definition of X and Y, any pair between them can only have one possible difference, namely e^t. Thus, there are at most $\min(|X|, |Y|)$ pairs between them.

Thus, we have the recurrence $a_n \le \max_{X+Y=n} (a_X + a_Y) + \min(X, Y)$. It thus suffices to show that if $Y \ge X$, $S(X+Y) - S(Y) \ge S(X) + X$, which expands to

$\sum_{k=Y}^{Y+X-1} s_2(k) \geq \sum_{k=0}^{X-1}(1+s_2(k))$. An exercise to the interested reader is to show this by strong induction on X.

Now it remains to evaluate $S(2023)$. By linearity of expectation, $S(2048)$ is equal to $2048 \cdot 11/2$. For every number from $2032 = 2048 - 16$ to 2048, 7 digits must be 1 and the remaining four each have a half chance of being 1, giving $S(2048) - S(2032) = 16 \cdot (7 + 4/2)$. Similarly $S(2032) - S(2024) = 8 \cdot (7 + 3/2)$, $S(2024) - S(2023)$ is just the number of ones in $2023 = 11111100111_2$ is 9. Thus the answer is

$$2048 \cdot 11/2 - 16 \cdot (7 + 4/2) - 8 \cdot (7 + 3/2) - 9 = 11043.$$

7. Show that for sufficiently large primes p, there is an Eulerian circuit on the complete graph with p vertices that does not contain any cycles of length at most 2023.

   SOLUTION. Take a generator $g \pmod p$ so that g is not equivalent to anything the form $-a/b$ for integers $a, b \leq 2023$. For big enough primes, such a g must exist as there are at most a constant number of such fractions.

   Now number the vertices with the residues mod p. For any residue $r \pmod p$, consider the sequence of vertices P_r formed by the multiples of r, starting from r and ending at $pr \equiv 0 \pmod p$. We claim that the concatenation of $P_1, P_g, P_{g^2}, \ldots$ works.

   Clearly no P_r contains a cycle. Thus, if a cycle were to exist, it would have to formed by the end of one P and the start of another. In other words, we would have $cg^{i+1} \equiv (p-d)g^i \pmod p$ for $c + d \leq 2023$. However, this is impossible by choice of g.