Berkeley Math Circle: Monthly Contest 1 Solutions

1. A straight ladder starts upright against a vertical wall, and slides down until it is horizontal such that the top is always along the wall and the bottom on the floor. What shape does its midpoint trace out?

   SOLUTION. The ladder always forms a right triangle with the wall whose hypotenuse has a fixed length. Since the length of the median to the hypotenuse in a right triangle is always half the hypotenuse, the distance from the midpoint of the ladder to the corner where the wall meets the floor remains fixed at half the length of the ladder. Thus, the path traced out by the midpoint is a $\boxed{\text{quarter circle}}$ whose center is the corner where the wall and floor meet and whose radius is half the length of the ladder.

2. Sofiya and Marquis play a game by taking turns. They form a circle with 2023 other people, and on each turn Sofiya or Marquis can remove one of their neighbors to the left or to the right from the circle. The person who removes the other player wins. If Sofiya starts, who has the winning strategy?

   SOLUTION. Note that there are an odd number of people in the circle beside Sofiya and Marquis, so Sofiya and Marquis divide the circle into two arcs, one with an even number of people and the other with an odd number of people. Sofiya's winning strategy will be to always remove a neighbor from the even side. This would leave Marquis with an odd number of people on either side of him (excluding Sofiya), so continuing this process would also leave Sofiya with an even side to remove a neighbor from. This process always leaves an odd number of people between Marquis and Sofiya whenever it is Marquis' turn, so he will never be able to remove Sofiya from the circle. This secures Sofiya's win.

3. Let a and b be integers. Show that 29 divides $3a+2b$ if and only if it divides $11a+17b$.

   SOLUTION. We give two solutions to this problem:

   First, observe that $11a + 17b \cong 23(3a + 2b) \pmod{29}$, so if $3a + 2b \cong 0 \pmod{29}$, then $11a + 17b \cong 0 \pmod{29}$. Similarly, if $11a + 17b \cong 0 \pmod{29}$, then since 23 is coprime to 29, we can invert $23 \pmod{29}$ and find that $3a + 2b \cong 0 \pmod{29}$.

   For another solution, we will find a linear combination $m(3a+2b)+n(11a+17b)$ which is a multiple of 29 for some integers m and n which are not themselves multiples of 29. This will imply that $m(3a + 2b)$ is a multiple of 29 if and only if $n(11a + 17b)$ is, and since we are choosing m and n to not be multiples of 29, it will follow that $3a + 2b$ is a multiple of 29 if and only if $11a + 17b$ is.

   Rewriting this, we want to find

   $$(3m + 11n)a + (2m + 17n)b = 29d$$

   for some multiple $29d$ of 29.

We can check the simplest case by setting $d = a + b$ and solving the linear system

$$3m + 11n = 29$$
$$2m + 17n = 29$$

Solving this system gives $m = 6, n = 1$, and since these are integer values, we're done.

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ from the integers to the integers satisfying

$$f(m + f(n)) - f(m) = n$$

for all integers $m, n \in \mathbb{Z}$.

SOLUTION. Adding $f(m)$ to both sides, we get $f(m+f(n)) = n+f(m)$. Swapping m and n gives $m + f(n) = f(n + f(m)) = f(f(m + f(n)))$. By fixing n and varying m, we can get $m + f(n)$ to be any integer x. Thus, $x = f(f(x))$.

Plugging in 1 for m and $f(x)$ for n to the original equation then gives $f(x) + f(1) = f(1 + f(f(x))) = f(1 + x)$, so $f(x)$ is linear, of the form cx for some constant c.

Plugging in 0 for m then gives $n = f(f(n)) - f(0) = c^2 n$, so $c^2 = \pm 1$. It is easily seen that both $\boxed{f(x) = x}$ and $\boxed{f(x) = -x}$ indeed work.

5. Find all positive integers n such that $n^4 - 27n^2 + 121$ is a prime positive integer.

SOLUTION. We can rewrite $n^4 - 27n^2 + 121$ as

$$(n^4 + 22n^2 - 121) - 49n^2 = (n^2 + 11)^2 - (7n)^2 = (n^2 + 7n + 11)(n^2 - 7n + 11).$$

For this to be prime, we would need $n^2 - 7n + 11 = 1$. Rearranging and factoring gives $(n-2)(n-5) = 0$, so $n = 2$ or $n = 5$. If $n = 2$, the expression equals 29, which is indeed prime, and if $n = 5$, the expression equals 71, which is also prime. Thus, our answer is $\boxed{n = 2, 5}$.

6. If $f(n, k)$ is the number of ways to divide the set $\{1, 2, \ldots, n\}$ into k nonempty subsets and m is a positive integer, find a formula for $\sum_{k=1}^{n} f(n, k)m(m - 1)(m - 2)\cdots(m - k + 1)$.

SOLUTION. We claim that the sum is equal to $\boxed{m^n}$. We note that m^n counts the number of ways to color n objects each with one of m different colors, so it suffices to show that the left side counts the same thing. We can consider cases based on how many different colors get used. If k colors are used, then there are $f(n, k)$ ways to choose how to split the objects into subsets based on color, then m ways to choose the color of the first subset, $m-1$ ways to colors the second one, and so on. Summing over k from 1 to n gives the desired sum.

(The numbers $f(n, k)$ are known as *Stirling numbers of the second kind*, and are usually denoted $S(n, k)$ or $\left\{{n \atop k}\right\}$.)

7. For points P, Q, R, let $E_{P,Q}(R)$ denote the ellipse with foci P and Q through R.

Points T, A, B, C are points on a line in that order. A ray from A intersects $E_{A,B}(T)$ at Q and $E_{A,C}(T)$ at P. Show that $E_{B,P}(A)$ intersects $E_{C,Q}(A)$ at two points.

SOLUTION. First of all, we claim that T is the tangency point of the B-excircle of ABQ to line AB. This is because

$$\frac{AQ + BQ - AB}{2} = \frac{AT + BT - AB}{2} = \frac{AT + AT}{2} = AT.$$

Similarly, T is also the tangency point of the C-excircle of ACP. Thus, as the center of the excircles both lie on the external angle bisector at A, these circles coincide, i.e. there is a circle tangent to all of TA, the ray, BQ, and CP.

Let BQ and CP intersect at R. We claim that $E_{B,P}(A)$ and $E_{C,Q}(A)$ are tangent at A and R.

They clearly intersect at A. For, $E_{B,P}(A)$, note that the excircles of APC and BRC are the same, so the semiperimeters of both are equal. Subtracting gives $BA + AP = BR + RP$, as desired. For $E_{C,Q}(A)$, note that the excircles of AQB and BRC are the same, so $2BT = AQ + QB + AB = BR + RC - BC$, i.e. $AC + AQ = RQ + RC$, as desired. Thus R lies on both ellipses as well.

Finally, the ellipses are tangent at A and R, because at each point the common tangents are the external angle bisectors.