Berkeley Math Circle: Monthly Contest 8 Solutions

1. If $P_1P_2 \ldots P_{100}$ is a regular 100-gon, what is the measure of the angle $\angle P_{20}P_2P_1$ in degrees?

   SOLUTION. Introduce the circumcircle of the polygon. The aforementioned angle cuts out an arc which is $\frac{81}{100}$ of that of the entire circle, i.e. which measures $\frac{81}{100} \cdot 360°$. By the inscribed angle theorem, this means an answer of $\frac{81}{100} \cdot 180° = 145.8°$.

2. For which positive integers n is $n^4 + 4$ equal to a prime number?

   SOLUTION. For $n = 1$ we get $1^4 + 4 = 5$, which works.

   For all other values of n, the key idea is that

   $$n^4 + 4 = n^4 + 4n^2 + 4 - 4n^2 = (n^2 + 2)^2 - (2n)^2 = (n^2 + 2n + 2)(n^2 - 2n + 2)$$

   which is the product of two integers greater than 1, and hence cannot be prime.

3. Find all real numbers x for which $\tan(x/2)$ is defined and greater than $\sin(x)$.

   SOLUTION. We claim that the answer is $x \in \left(k\pi - \frac{\pi}{2}, k\pi\right)$ for $k \in \mathbb{Z}$.

   The statement is never true for $x/2$ a multiple of $\frac{\pi}{2}$ because $\tan x/2$ is either undefined or equal to $0 = \sin x$. Thus $x/2$ is not a multiple of $\frac{\pi}{2}$, so $\cos^2(x/2) > 0$, and

   $$\begin{aligned} \tan(x/2) &\stackrel{?}{>} \sin(x) \\ \cos^2(x/2)\tan(x/2) &\stackrel{?}{>} \cos^2(x/2)\sin(x) \\ 2\cos(x/2)\sin(x/2) &\stackrel{?}{>} 2\cos^2(x/2)\sin(x) \\ \sin(x) &\stackrel{?}{>} (1 + \cos(x))\sin(x) \\ 0 &\stackrel{?}{>} \cos(x)\sin(x) \\ 0 &\stackrel{?}{>} \sin(2x), \end{aligned}$$

   as desired.

4. Aerith and Bob take turns picking a nonnegative integer, each time changing exactly one digit from the other's last number. The first person to pick a number that (s)he picked before loses. If Aerith goes first, and both play optimally, who wins?

   (Note: There are no leading zeroes, except in the number 0 itself. For instance, if one person picks 2020, the other could respond by picking 0020 = 20, however the reverse does not hold.)

SOLUTION. Bob wins. One winning strategy for Bob is: each time Aerith picks an even number, add one, and each time Aerith picks an odd number, subtract one. This only changes the last digit, since there are no carryovers in either case.

Bob would only get into a situation where he repeated an even number if Aerith had repeated the succeeding number twice, and similarly he would only need to repeat an odd number if Aerith had picked the preceding number twice. In either case, Aerith would have already lost.

5. A set S of irrational real numbers has the property that among any subset of five numbers in S, one can find two with irrational sum. How large can $|S|$ be?

SOLUTION. The answer is $|S| \le 8$. An example is $S = \left\{n \pm \sqrt{2} \mid n = 1, 2, 3, 4\right\}$ (and any of its subsets).

In general, construct a graph with vertex set S in which we join two numbers with rational sum. We claim this graph is bipartite; indeed if $a_1 + a_2$, $a_2 + a_3$, $\dots$, $a_n + a_1$ are all rational for some odd n, solving the resulting system of equations gives a_1, $\dots$, a_n all rational numbers.

Accordingly we may 2-color S. If $|S| \ge 9$, then we may find a set of five numbers with rational sum, as desired.

6. Show that if n is a positive integer for which $2 + 2\sqrt{1 + 12n^2}$ is an integer, then it is a perfect square.

SOLUTION. Let the value of this expression be $2k$, say. Then we may conclude that

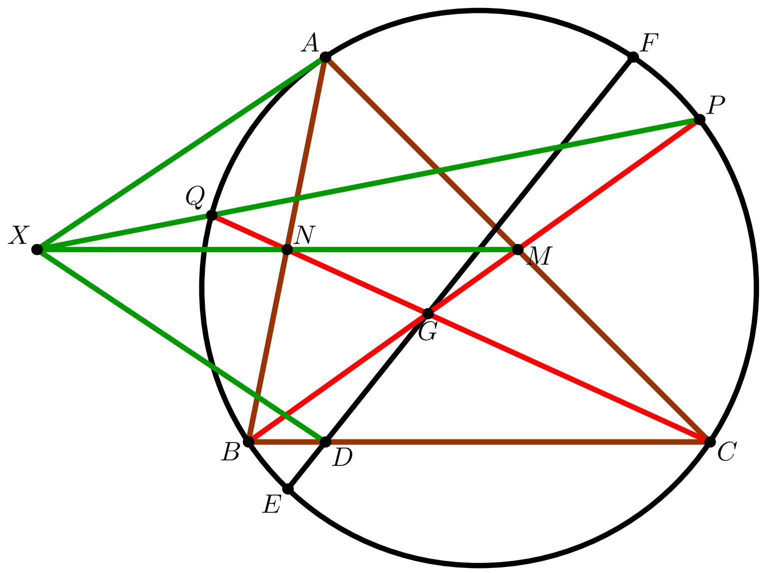
$$12n^2 + 1 = (k-1)^2 \implies 12n^2 = k(k-2).$$

Obviously k is even and hence $\gcd(k, k-2) = 2$. So we may write this factorization as $2x^2$ and $6y^2$ in some order.

If $k = 2x^2$ then $2k = 4x^2$ and we are done. But otherwise $k - 2 = 2x^2$ and $k = 6y^2$ which gives $3y^2 - 1 = x^2$, and this is a contradiction modulo 3. This solves the problem.

7. Let ABC be an acute scalene triangle with centroid G. The rays BG and CG meet the circumcircle of ABC again at points P and Q. Let D denote the foot of the altitude from A to BC. Suppose ray GD meets the circumcircle of ABC again at E. Show that the circumcircle of triangle ADE lies on line PQ.

SOLUTION. Let M and N be the midpoints of CA and AB. By Pascal's theorem on $AABPQC$, we find that the tangent to the circumcircle at A, the line MN, and the line PQ are concurrent at a single point X.



Our claim is that X is the desired circumcenter.

First note that $XA = XD$ simply because MN is the perpendicular bisector of segment AD.

To proceed further we'll introduce point F, the intersection of ray DG with the circumcircle. Let's assume for simplicity that $\angle B > \angle C$ (the other case is analogous). Owing to the negative homothety at G mapping the nine-point circle to the circumcircle, we find that $ABCF$ is an isosceles trapezoid. Thus

$$\angle AED = \angle AEF = \angle B - \angle C = \angle ANM - \angle XAB = \angle AXM = \frac{1}{2}\angle AXD$$

and this completes the proof.