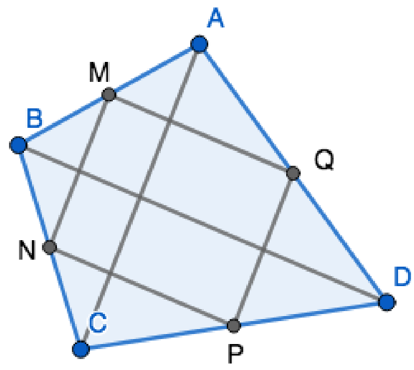
Berkeley Math Circle: Monthly Contest 7 Solutions

1. Given a quadrilateral $ABCD$, show that the midpoints of its four edges form the vertices of a parallelogram.

   SOLUTION.



   Let M, N, P, Q be the midpoints of AB, BC, CD, and DA, respectively. Then MN is the midline of $\triangle ABC$ opposite AC, so it is parallel to AC and of length $\frac{1}{2}AC$. Similarly, PQ is the midline of $\triangle ACD$, so $PQ \parallel AC$ and $PQ = \frac{1}{2}AC$. Thus, the opposite sides MN and PQ are equal and parallel, and similarly, NP and QM are equal and parallel. Thus, $MNPQ$ is a parallelogram.

2. Let x, y, z be nonzero real numbers such that the equations

$$x + \frac{1}{y} = y + \frac{1}{x}$$
$$y + \frac{1}{z} = z + \frac{1}{y}$$
$$z + \frac{1}{x} = x + \frac{1}{z}$$

   all hold. Show that two of the three variables must be equal.

   SOLUTION. The equation $x + \frac{1}{y} = y + \frac{1}{x}$ rearranges as

$$0 = x^2 + \left(\frac{1}{y} - y\right)x - 1 = \left(x + \frac{1}{y}\right)(x - y)$$

   so either $x = y$ or $x = -1/y$.

   By applying similar logic, we conclude that $y = z$ or $y = -1/z$.

   If we assume $x \neq y$, $y \neq z$, we thus see that $x = z$ follows. (Note the third equation was never used.)

3. There are m friends with n cupcakes each weighing 1 ounce. They wish to split the cupcakes equally by dividing each cupcake into some number of parts, and allocating some parts to each person.

   a) Assume $m = 3$ and $n = 5$. Show that they may divide the cupcakes with all pieces being larger than $\frac{1}{3}$ ounces.

   b) Assume $m = 5$ and $n = 3$. Show that they may divide the cupcakes with all pieces being larger than $\frac{1}{5}$ ounces.

   SOLUTION.

   a) Each person needs $\frac{5}{3} = \frac{20}{12}$ of a cupcake. If they cut four of the cupcakes into $\frac{5}{12}$ and $\frac{7}{12}$ and the last cupcake in half, then one person can take the four $\frac{5}{12}$ pieces, giving them
   $$4 \cdot \frac{5}{12} = \frac{20}{12}$$
   and the other two can each take two $\frac{7}{12}$ pieces and one $\frac{1}{2} = \frac{6}{12}$ piece, giving them
   $$\frac{7}{12} + \frac{7}{12} + \frac{6}{12} = \frac{20}{12}$$
   as well.

   b) Each person needs $\frac{3}{5} = \frac{12}{20}$ of a cupcake. We can divide two of the cupcakes into $\frac{6}{20}$, $\frac{7}{20}$ and $\frac{7}{20}$ pieces and the last cupcake into four $\frac{1}{4} = \frac{5}{20}$ pieces. Then four of the people can get one $\frac{7}{20}$ piece and one $\frac{5}{20}$ piece and the fifth person can get the two $\frac{6}{20}$ pieces.

4. Eight friends, Aerith, Bob, Chebyshev, Descartes, Euler, Fermat, Gauss, and Hilbert, bought tickets for adjacent seats at the opera. However when they arrived they mixed up their seats:

   - Bob sat in his assigned seat,
   - Chebyshev sat two seats to the right of Gauss' assigned seat,
   - Descartes sat one seat to the left of Fermat's assigned seat,
   - Euler sat four seats to the left of Hilbert's assigned seat,
   - Fermat sat five seats to the right of Descartes' assigned seat,
   - Gauss sat one to the right of Euler's assigned seat,
   - Hilbert sat three seats to the left of Aerith's assigned seat.

   In whose seat did Aerith sit?

   SOLUTION. Number the seats 1 through 8 and let $a, \ldots, h$ be the seat assignments. Let A be the seat occupied by Aerith. As each seat is assigned to exactly one person we must have $a + \cdots + h = 1 + \cdots + 8$. As each seat is occupied by exactly one person we must have
   $$\begin{aligned} 1 + \cdots + 8 &= A + b + (g + 2) + (f - 1) + (h - 4) + (d + 5) + (e + 1) + (a - 3) \\ &= A + (a + b + d + e + f + g + h). \end{aligned}$$
   Thus $A = c$, so Aerith occupies Chebyshev's seat.

5. Let n be a positive integer which also divides $2^n - 1$. Show that $n = 1$.

SOLUTION. Assume not and let p be the smallest prime divisor of n. We have $p \mid 2^n - 1$, and also $p \mid 2^{p-1} - 1$ by Fermat's little theorem. By using the classical fact that $\gcd(2^x - 1, 2^y - 1) = 2^{\gcd(x,y)} - 1$, we conclude p divides $2^{\gcd(p-1,n)} - 1$.

But since p is the smallest divisor of n, it follows $\gcd(p-1, n) = 1$. So p divides $2^1 - 1 = 1$ which is absurd.

6. Let ABC be an acute triangle with circumcenter O, incenter I, orthocenter H. If $OI = HI$, what are the possible values of the angles of triangle ABC?

SOLUTION. Answer: this occurs if and only if some angle is 60 degrees.

One direction is immediate; if $\angle A = 60^\circ$ then $BHOIC$ are cyclic since $\angle BHC = \angle BIC = \angle BOC = 120^\circ$.

For the other direction, note that we have an "SSA congruence" of triangles AIH and AIO. Consequently, either A lies on the circle (OIH) or $\triangle AIH \cong \triangle AIO$. In latter case, $AH = AO$, but since it's known that $AH = 2AO\cos A$, it follows that $\cos A = \frac{1}{2} \implies \angle A = 60^\circ$.

Now it's impossible for $A, B, C \in (OIH)$ since the points A, B, C, O are not concyclic. Thus some angle must be 60°.

7. Prove that if positive real numbers x, y, z have sum 1, then

$$\frac{x}{x+yz} + \frac{y}{y+zx} + \frac{z}{z+xy} \le \frac{2}{1-3xyz}.$$

SOLUTION. The idea is to use the identity

$$\frac{x}{x+yz} = \frac{x}{x(x+y+z)+yz} = \frac{x}{(x+y)(x+z)} = \frac{x(y+z)}{(x+y)(y+z)(z+x)}.$$

So the left-hand side is *exactly* equal to

$$\frac{2(xy+yz+zx)}{(x+y)(y+z)(z+x)} = \frac{2(xy+yz+zx)(x+y+z)}{(x+y)(y+z)(z+x)}.$$

We let $k = (xy+yz+zx)(x+y+z)$. On the other hand, we claim that $k \le 1/3$, which is sufficient. Indeed, to prove $k \le 1/3$ it suffices to prove that

$$xy + yz + zx \le \frac{(x+y+z)^2}{3}$$

which is a direct consequence of Cauchy-Schwarz inequality.

On the other hand, the left-hand side equals, exactly,

$$\frac{2k}{k-xyz} = 2 + \frac{2xyz}{k-xyz}.$$

So simply noting $k \le 1/3$ implies the desired conclusion, since this is a decreasing function of k.