Berkeley Math Circle: Monthly Contest 6 Solutions

1. Find all nonzero real numbers x such that
$$x^2 + \frac{36}{x^2} = 13.$$

SOLUTION. Multiplying through by x^2 and moving all terms to the left gives
$$x^4 - 13x^2 + 36 = 0.$$
We can factor this as
$$(x^2-4)(x^2-9) = (x-2)(x+2)(x-3)(x+3) = 0.$$
Thus, the solutions are $x = \pm 2$ and $x = \pm 3$.

2. Prove or disprove the following assertion: given 3 noncollinear points in the plane, the disk with the smallest radius which contains all three points is the circle passing through all three points.

SOLUTION. The assertion is false. Consider the points $A = (-\sqrt{3}, 0)$, $B = (\sqrt{3}, 0)$, $C = (0, 1)$. They are contained in the circle with diameter AB, which has radius $\sqrt{3}$, but the circumcircle of the triangle ABC has radius 2 (centered at $O = (0, -1)$).

3. For any positive integer $n \geq 3$, show that you can write 1 as a sum of n fractions with numerator 1 and different denominators.

SOLUTION. For $n = 3$, we can write
$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$
For $n = 4$, we can write
$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18}.$$
We claim that we can continue this pattern by breaking the smallest fraction into two smaller fractions, such that the smallest fraction always has even denominator. We can show this inductively. Assume this is true for some n, where the smallest fraction is $\frac{1}{2m}$. Then we can break the $\frac{1}{2m}$ into
$$\frac{1}{2m} = \frac{1}{3m} + \frac{1}{6m},$$
which creates a sum of $n+1$ distinct fractions with numerator 1, and the largest denominator is still even. This completes our induction.

4. Aerith and Bob are playing “Not Quite Tic-Tac-Toe”, in which an X is written on a line, and each player takes tuns adding either an X or an O (their choice) to the end of the line. Aerith goes first, and the goal is to avoid a sequence of three evenly spaced X’s or O’s; the first person to do so loses.

   For instance, if the letters on the line are currently XOXXOX, Bob is forced to write down O in order to avoid $\underline{\text{X}}$OX$\underline{\text{X}}$OX$\underline{\text{X}}$. Aerith would then lose, as both XOX$\underline{\text{X}}$O$\underline{\text{X}}$O$\underline{\text{X}}$ and X$\underline{\text{O}}$XX$\underline{\text{O}}$XO$\underline{\text{O}}$ are losing positions.

   Assuming that both of them play optimally after Aerith’s first move, who wins if she starts by putting down a second X on the line next to the initial one? What if she starts with an O?

   SOLUTION. If Aerith opens with an X, she can win simply by repeating Bob’s last letter on each turn. This forces the game to end up at XXOOXXOO, at which point Bob will lose on his next move.

   Otherwise, we claim that Aerith can still win, and show this by casework:

   - Case 1: The sequence starts with XOX.

     In this case, Aerith plays X giving XOXX, forcing Bob to play O giving XOXXO. Then Aerith responds by playing O, giving XOXXOO, forcing Bob to lose on his next turn.

   - Case 2: The sequence starts with XOO.

     In this case Aerith is forced to play X next, giving XOOX. We get the following subcases after Bob’s next move:

     – **Case 2a: The sequence starts with XOOXO**. In this case Aerith plays O giving XOOXOO, forcing Bob to lose on his next turn.

     – **Case 2b: The sequence starts with XOOXX**. In this case Aerith is forced to play O next, giving XOOXXO. Bob is then forced to play O, giving XOOXXOO, and forces Aerith to play X to get XOOXXOOX. At this point Bob loses on his next turn.

5. Does there exist a function f from the positive integers to itself, such that for any positive integers a and b, we have $\gcd(a, b) = 1$ if and only if $\gcd(f(a), f(b)) > 1$ holds?

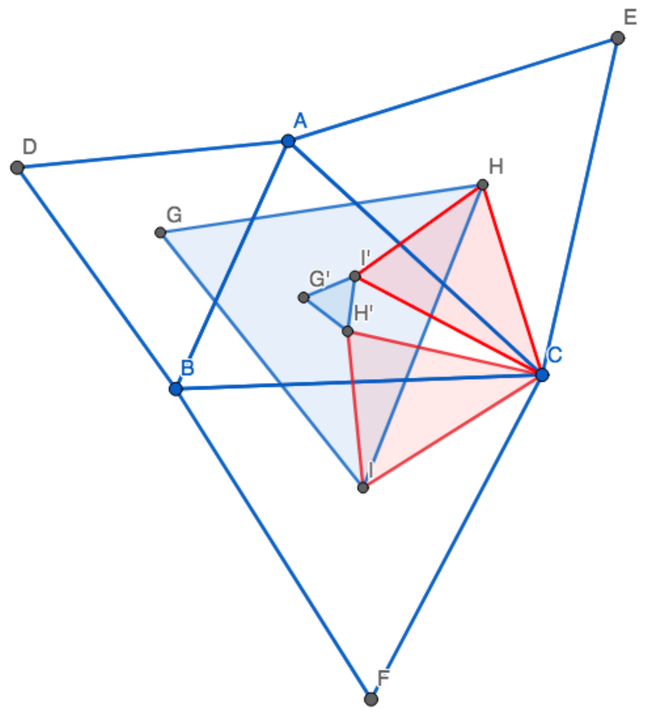
   SOLUTION. The answer is no. Assume that f satisfies the hypothesis. Let k denote the number of distinct primes dividing $f(1)$. For every integer e, the number $f(2^e)$ shares some prime factor with $f(1)$. So among $f(2)$, $f(4)$, $\dots$, $f(2^{k+1})$ two of them have the same shared prime factor, which is impossible.

6. Equilateral triangles $\triangle ABD$, $\triangle ACE$, and $\triangle BCF$ are drawn outside $\triangle ABC$ on each of its sides, with centers G, H, and I, respectively.

   a) Prove that $\triangle GHI$ is equilateral.

   b) Let G', H', and I' be the reflections of G, H, and I across AB, BC, and CA, respectively. Prove that $\triangle GHI$ and $\triangle G'H'I'$ have the same circumcenter.

SOLUTION.



a) Let O be the intersection of circles ABD and ACE (besides A). Since opposite angles in a cyclic quadrilateral sum to $180°$, $\angle AOB = \angle AOC = 120°$, so $\angle BOC = 120°$ as well. Thus, quadrilateral $BOCF$ is also cyclic. Since A and O are the two intersection points of circles with centers G and H, $AO \perp GH$. Similarly, $BO \perp GI$. Since $\angle AOB = 120°$, $\angle GHI = 60°$. similarly, $\angle HGI = \angle HIG = 60°$, so $\triangle GHI$ is equilateral.

b) We can use similar logic to show that $\triangle G'H'I'$ is equilateral as well (since G', H', and I' are the centers of equilateral triangles drawn in the opposite direction on each side of $\triangle ABC$).

Next, we claim that
$$\triangle CHI' \cong \triangle CH'I \sim \triangle ABC.$$
We can see that CH' is a $60°$ rotation of CH about point C, and CI is a $60°$ rotation of CI' about C in the same direction. Also, $\triangle ACH$ and $\triangle BCI$ are isosceles triangles with base angle $30°$. Thus,
$$CH = CH' = AC/\sqrt{3},$$
$$CI' = CI = BC/\sqrt{3},$$
$$\angle HCI' = \angle ACB = \angle H'CI.$$
The desired similarly and congruence follows by SAS, and we conclude that
$$IH' = I'H = AB/\sqrt{3} = GG'.$$
By similar logic,
$$HH' = IG' = G'I.$$

Thus by SSS,

$$\triangle IHH' \cong \triangle HI'G \cong \triangle GG'I.$$

Thus, the vertices of $\triangle G'H'I'$ are positioned symmetrically with respect to the sides of $\triangle GHI$, and it follows that the two equilateral triangles have the same center.

7. Do there exist positive integers $a_1, \ldots, a_{100}$ such that for each $k = 1, \ldots, 100$, the number $a_1 + \cdots + a_k$ has exactly a_k divisors?

SOLUTION. Answer: yes.

The idea is to define the partial sums instead as follows. Let $d(n)$ denote the divisor function. Let s_N be suitably large, then define by downwards recursion

$$s_n = s_{n+1} - d(s_{n+1})$$

with N set such that $s_0 = 1$. Then $s_k = a_1 + \cdots + a_k$ works fine.

All we need is to ensure this sequence stays positive. But $x - d(x) \geq x/2$ for $x \geq 8$, so it's enough to take $s_N = 2^{103}$.