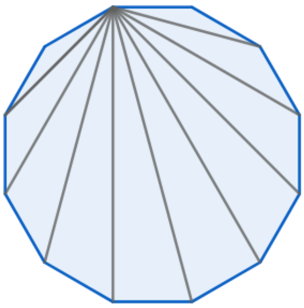
Berkeley Math Circle: Monthly Contest 5 Solutions

1. A polygon is *regular* if all its sides and angles are the same. Find the measure of each angle in a regular dodecagon (12-sided polygon).

   SOLUTION.

   A regular dodecagon can be divided into 10 triangles, as shown above. The sum of the angles in each triangle is $180°$, so the sum of the angles in the dodecagon is $10 \cdot 180° = 1800°$. Since it is regular, all the angles are the same, so we divide by 12 to get $1800°/12 = 150°$ for each angle.

2. Prove that if x is an positive real number such that $x + x^{-1}$ is an integer, then $x^3 + x^{-3}$ is an integer as well.

   SOLUTION. This follows from the identity

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right).$$

3. Show that for any positive integers a, b, and c,

$$abc \operatorname{lcm}(a, b, c) \geq \operatorname{lcm}(b, c) \operatorname{lcm}(c, a) \operatorname{lcm}(a, b),$$

   where lcm denotes the least common multiple.

   SOLUTION. It suffices to show that the left hand side is a multiple of the right, i.e., that for any prime p, the exponent of p in the prime factorization of the left hand side is at least that for the right.

   Let the exponent of p in a, b, c be x, y, z, respectively. WLOG, assume $x \geq y \geq z$. The exponent of p in the left hand side is then $x + y + z + \max(x, y, z) = 2x + y + z$, and that in the right is $\max(y, z) + \max(x, z) + \max(x, y) = y + 2x$, so the claim follows.

4. Aerith and Bob play rounds of pool. At some point Bob had won more rounds than Aerith, but now Aerith has won 85% of their rounds. Show that at some point, Aerith had won exactly 75% of their rounds.

SOLUTION. At any time, let a and b represent the respective number of rounds won by Aerith and Bob. Now, after n games, $a-3b = n(85\%-3\cdot 15\%) > 0$. However, at some point in the past b was greater than a, so $a-3b$ was negative. Consider the first subsequent moment wwhen $a-3b$ was non-negative. At that moment,

$$(a-1)-3b < 0 \leq a-3b,$$

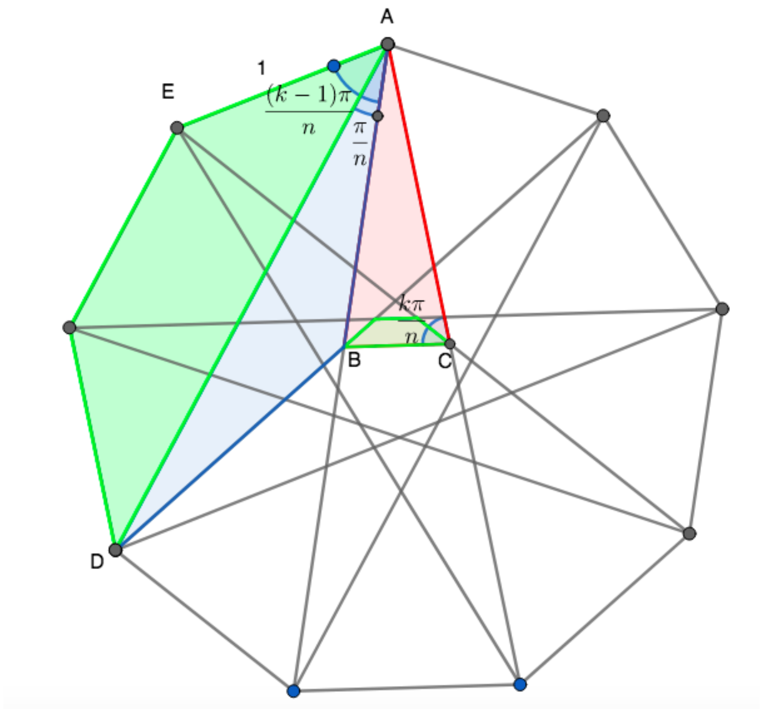
so as a and b were integers, $a-3b=0$ and $a = 75\% \cdot (a+b)$, as desired.

5. In a regular n-gon, all the diagonals are drawn, forming smaller regular n-gons inside. If the outer regular n-gon has side length 1, show that the kth largest regular n-gon formed has side length

$$\frac{\cos(k\pi/n)}{\cos(\pi/n)}$$

(where the original regular n-gon is the 1st largest).

SOLUTION. Draw a diagonal of the small n-gon to form the red triangle $\triangle ABC$ as shown in the diagram, where A is a vertex of the large n-gon and B and C are the vertices of the small n-gon closest to A on the two diagonals from A. Also draw the diagonal AD of the large n-gon such that $\angle DAB = \pi/n$.



Let E be the vertex of the large n-gon adjacent to A on the same side of A as D. Then $\angle EAB = (k-1)\pi/n$, and since the angles of the n-gon are $(n-2)\pi/n$,

$$\angle BAC = \frac{(n-2)\pi}{n} - 2 \cdot \frac{(k-1)\pi}{n} = \frac{(n-2k)\pi}{n},$$

so since $\triangle ABC$ is isosceles, the base angle is

$$\angle ABC = \angle ACB = \frac{k\pi}{n}.$$

Thus we have

$$\frac{BC}{AB} = 2\cos(k\pi/n).$$

Since $\angle BAD = \pi/n$,

$$\frac{AB}{AD} = \frac{1}{2\cos(\pi/n)} \implies \frac{BC}{AD} = \frac{\cos(k\pi/n)}{\cos(\pi/n)}.$$

But the green shapes are similar, so the ratio of the side length of the small n-gon to the side length of the large n-gon is equal to the ratio BC/AD. This proves our claim.

6. Can $x^{2020} - 8$ be written as the product of two nonconstant polynomials with integer coefficients?

SOLUTION. The answer is no. Indeed, suppose we had $x^{2020} - 8 = f \cdot g$. Since every complex root of f has absolute value $\sqrt[2020]{8}$, and the product of these roots is an integer (by Vieta formula), we conclude that

$$8^{\frac{\deg f}{2020}} = 2^{\frac{3 \deg f}{2020}}$$

must be an integer, which could only occur if $\deg f \in \{0, 2020\}$.

7. Prove that there exists infinitely many integers n such that $n^4 + 2020$ has a prime divisor larger than $2n$.

SOLUTION. Note that if p is a prime that divides $x^4 + 2020$ for some x, then we can find n such that $p \mid n^4 + 2020$ and $n < p/2$. Indeed, just take n to be the remainder when $p - x$ or x is divided by p, whichever is smaller.

Suppose S is a set of good integers satisfying the property; we show how to find one more. By so-called Schur's theorem, there are infinitely many primes p that divide $x^4 + 2020$ for some x. Take such a prime p large enough, not dividing $s^4 + 2020$ for any $s \in S$; then taking the n generated in the previous paragraph, we get a new pair (p, n), with $n \notin S$ guaranteed.