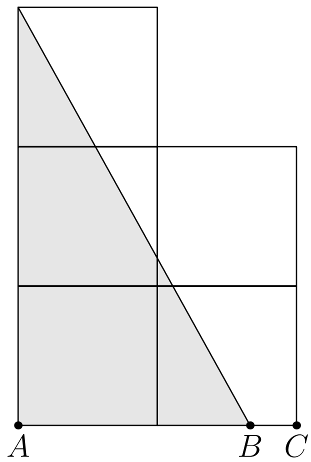
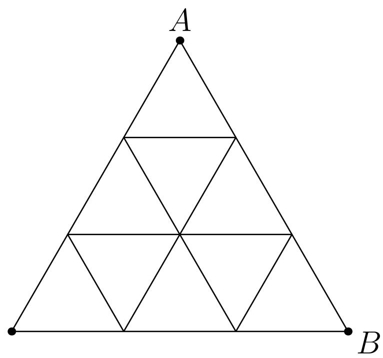
Berkeley Math Circle: Monthly Contest 4 Solutions

1. The figure below consists of five congruent squares. If the area of the shaded triangle equals the area outside the shaded triangle, calculate the ratio AB/BC.



SOLUTION. Let s be the side length of the square. As the total area of the figure is $5s^2$, the triangle should have area $5s^2/2$. Here AB is a base of the triangle and the height is $3s$, hence $AB = 5s/3$. Since $AC = 2s$, we get $BC = s/3$ and we obtain the final answer $AB : BC = 5$.

2. Suppose you have an equilateral triangle divided into 9 smaller equilateral triangles as shown, with the bottom side horizontal. Starting from the top corner labeled A, you must walk to the bottom right corner labeled B, and are only allowed to take steps along the edges down to the left, down to the right, or horizontally to the right. Determine the number of possible paths.



SOLUTION. The number of paths from the top vertex to itself or to any vertex on the left side of the triangle is 1. For each other vertex, the number of paths can be found by adding the number of paths to the vertex directly to its left and to the two vertices above it. Thus, the following triangle of numbers tells us the number of paths to each vertex, where each number represents a vertex:

$$\begin{array}{cccc} & & 1 & \\ & 1 & 2 & \\ 1 & 4 & 6 & \\ 1 & 6 & 16 & 22. \end{array}$$

The bottom right number is 22, so the number of paths to the bottom right corner is 22.

3. Let gcd mean the greatest common divisor of two numbers and lcm their least common multiple. Suppose the three numbers A, B, C satisfy

$$\gcd(A,B) = 2, \quad \text{lcm}(A,B) = 60$$
$$\gcd(A,C) = 3, \quad \text{lcm}(A,C) = 42.$$

Determine the three numbers.

SOLUTION. From the given information, A must be a multiple of 2 and 3, and thus a multiple of $LCM(2,3) = 6$. It also must be a factor of 60 and 42, and thus a factor of $GCD(60,42) = 6$. The only possibility is $A = 6$.

Since $LCM(A,B)$ is divisible by 5 but A is not, B must be divisible by 5. Similarly, since $LCM(A,B)$ is divisible by $2^2 = 4$ but A is not, B must also be a multiple of 4 and thus a multiple of 20. B cannot be 60 or we would have $GCD(A,B) = 6$, thus $B = 20$.

Finally, since $LCM(A,C)$ is a multiple of 7 but A is not, C must be divisible by 7. Also, C is divisible by 3 since $GCD(A,C) = 3$. Thus, C is a multiple of 21, and we cannot have $C = 42$ or else $GCD(A,C)$ would be 6. Thus, $C = 21$, giving the solution $A = 6, B = 20, C = 21$.

4. Aerith timed how long it took herself to solve a BMC monthly contest. She writes down the elapsed time as days:hours:minutes:seconds, and also simply as seconds. For example, if she spent 1,000,000 seconds, she would write down 11:13:46:40 and 1,000,000.

Bob sees her numbers and subtracts them, ignoring punctuation; in this case he would get

$$11134640 - 1000000 = 10134640.$$

What is the largest number that always must divide his result?

SOLUTION. Say that Aerith took d days, h hours, m minutes, and s seconds. Bob would then get

$$\begin{aligned}\Delta &= (100^3 d + 100^2 h + 100m + s) - (24 \cdot 60 \cdot 60d + 60 \cdot 60h + 60m + s) \\ &= (100^3 - 24 \cdot 60 \cdot 60)d + (100^2 - 60 \cdot 60)h + (100 - 60)m + (1 - 1)s \\ &= (2^6 \cdot 5^6 - 2^7 \cdot 3^3 \cdot 5^2)d + (2^4 \cdot 5^4 - 2^4 \cdot 3^2 \cdot 5^2)h + 40m \\ &= \left(2^3 \cdot 5\right)\left[(2^3 \cdot 5^5 - 2^4 \cdot 3^3 5)d + (2 \cdot 5^3 - 2 \cdot 3^2 \cdot 5)h + m\right],\end{aligned}$$

so $2^3 \cdot 5 = 40$ must divide his number. However, no larger number works, because if Aerith spent 60 seconds = 1:00, Bob would get $100 - 60 = 40$ seconds.

5. Are there integers a and b for which $a^2 = b^{15} + 1004$?

SOLUTION. The answer is no. Work modulo 31: the equation reads $a^2 \equiv b^{15} + 12$. (The choice of 31 is motivated by $2 \cdot 15 + 1 = 31$.) We must have $b^{15} \in \{-1, 0, 1\} \pmod{31}$ since the square of $b^{15} \pmod{31}$ is $b^{30} \pmod{31}$ which is always 0 or 1.

However, one can exhaustively check that none of 11, 12, 13 are quadratic residues modulo 31. So this concludes the proof.

6. In a convex n-sided polygon, all the diagonals are drawn and no three of them pass through a point. Find a formula for the number of regions formed inside the polygon.

SOLUTION. To start with, there is one region inside the n-gon. A new region is created each time a new diagonal is drawn or two diagonals intersect. The number of diagonals is $\binom{n}{2} - n$, since each diagonal corresponds to a pair of non-adjacent vertices. The number of intersections of two diagonals is $\binom{n}{4}$, since each set of four vertices determines a unique pair of intersecting diagonals. Thus, the total number of regions formed is

$$1 + \binom{n}{2} - n + \binom{n}{4}.$$

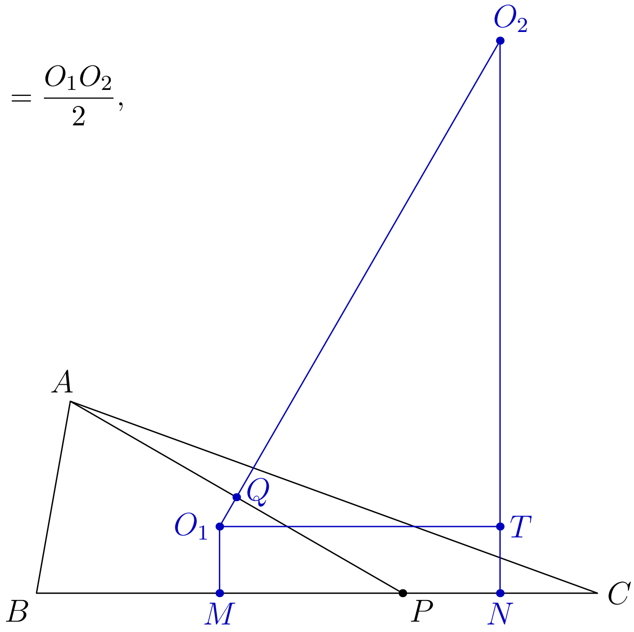
7. Let P be a point on segment BC of triangle $\triangle ABC$. Let O_1 and O_2 be the respective circumcenters of $\triangle ABP$ and $\triangle ACP$. Given $BC = O_1O_2$, show that some angle of $\triangle ABC$ is more than $75°$.

SOLUTION. WLOG, assume O_1 is at least as close to BC as is O_2. Let M, N, and Q be the respective midpoints of $\overline{BP}, \overline{CP}$, and $\overline{AP}$, and let T be the foot of the altitude from O_1 to $\overline{O_2N}$.

As $\overline{O_1M}$ and $\overline{O_2N}$ are the perpendicular bisectors of $\overline{BP}$ and $\overline{PC}$, respectively, O_1TNM is a rectangle, so

$$O_1T = MN = MP + PN = \frac{BP}{2} + \frac{CP}{2} = \frac{BC}{2} = \frac{O_1O_2}{2},$$

and $\triangle O_2O_1T$ is a respective $30^\circ - 60^\circ - 90^\circ$ triangle. Therefore, looking at the sum of the angles of quadrilateral $NPQO_2$, $\triangle ACP$, and $\triangle ABC$, we find



- $\angle NPQ = 360^\circ - \angle PQO_2 - \angle QO_2N - \angle O_2NP = 360^\circ - 90^\circ - 30^\circ - 90^\circ = 150^\circ$,
- $\angle ACP = 180^\circ - \angle CPA - \angle PAC < 180^\circ - \angle CPA = 180^\circ - 150^\circ = 30^\circ$, and
- $\angle A + \angle B = 180^\circ - \angle C > 180^\circ - 30^\circ = 150^\circ$.

This is impossible unless either $\angle A$ or $\angle B$ is $> 75^\circ$, as desired.