Berkeley Math Circle: Monthly Contest 3 Solutions

1. Find four different ways to write 24 using the numbers 4, 5, 7, 8 (each exactly once). You can use addition, subtraction, multiplication, division, and parentheses. Reordering numbers that are added or multiplied together does not count as different.

   SOLUTION. Four ways to do this are:

   - $(5 + 7) \times 8/4$
   - $(8 - 7 + 5) \times 4$
   - $4 + 5 + 7 + 8$
   - $(7 - 5) \times (8 + 4)$.

2. Let x, y, z be positive real numbers with $x + 2y + 3z = 1$. Find the maximum value of $\min(2xy, 3xz, 6yz)$.

   SOLUTION. By AM-GM,

$$\sqrt{2xy} \le \frac{x + 2y}{2},$$
$$\sqrt{3xz} \le \frac{x + 3z}{2},$$
$$\sqrt{6yz} \le \frac{2y + 3z}{2}.$$

   Adding gives

$$\sqrt{2xy} + \sqrt{3xy} + \sqrt{6yz} \le x + 2y + 3z = 1.$$

   Thus, $\min(\sqrt{2xy}, \sqrt{3xz}, \sqrt{6yz}) \le \frac{1}{3}$, so $\min(2xy, 3xz, 6yz) \le \frac{1}{9}$. This is achievable by setting $x = \frac{1}{3}, y = \frac{1}{6}, z = \frac{1}{9}$, so the minimum is $\boxed{\frac{1}{9}}$.

3. Suppose chameleons are either red, green, or blue, and whenever two of them of different colors meet, they both change to the third color. Otherwise, they don't change colors. Suppose you start with 12 red chameleons, 13 green chameleons, and 14 blue chameleons. Is it possible that at some point all the chameleons will be red?

   SOLUTION. This is not possible. Assign red chameleons the number 0, green chameleons 1, and blue chameleons 2. Then the mod 3 sum of all the chameleons remains invariant, since whenever two of them meet the sum does not change. The starting sum is

$$12 \cdot 0 + 13 \cdot 1 + 14 \cdot 2 \equiv 0 + 1 + 2 \cdot 2 \equiv 2 \pmod 3.$$

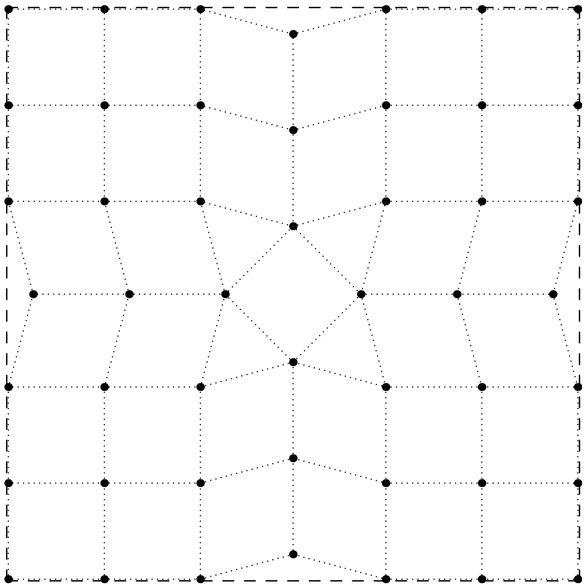
   If all the chameleons were red, the sum would be 0, so this can't happen.

4. a) Can Aerith put 48 points in the interior of a 6×6 square such that no two are at most one unit apart?

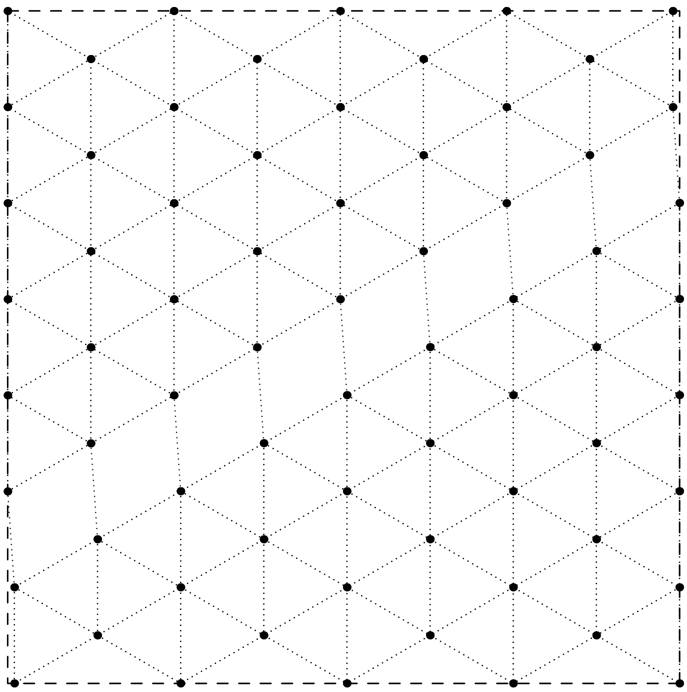
   b) Can Bob put 68 points in the interior of a 7×7 square such that no two are at most one unit apart?

SOLUTION.

a) Yes; the diagram shows such an arrangement. Note that the marked distances are slightly more than 1 and that all points lie within the boundary of the square and not on the boundaryitself .



b) Yes, as shown in the diagram.



5. What is the smallest nonnegative integer n for which $\frac{(20n)!}{(n!)^{21}}$ is not an integer?

SOLUTION. Let $f(n)$ be $\frac{(20n)!}{(n!)^{21}}$.

The number of ways to partition $20n$ objects into 20 unordered sets of size n is $g(n) = \frac{(20n)!}{20! \cdot (n!)^{20}}$, as there are $(20n)!$ ways to permute the objects, $n!$ ways to permute elements within each setand 20! ways to permute the sets themselves. Because this is a divisor of $f(n)$ for $n \leq 20$, all $n \leq 20$ work.

For $n = 21, 22$, one can check that $f(n)$ is still an integer. However, at $n = 23$, 23^{21} divides the denominator of $f(n)$ but not the numerator. Thus the answer is 23.

6. Bob is thinking of a positive integer under 100. Aerith wants to figure out his number, and can ask him seven yes or no questions about it. However, if Bob says no four times, he will feel too sad to answer any more questions. Show that Aerith can always figure out his number.

SOLUTION. Let N be Bob's integer. $\binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7} = 99$, so we can biject the first 99 integers to subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ of size at least 3; let S be one such bijection. Aerith's n^{th} question should then be "Is $n \in S(N)$?". Because the answer is "no" at most four times, even if Bob were to stop answering her questions, she would know all the remaining answers would be "yes". Thus, Aerith can always figure out $S(N)$ and thus N.

7. Suppose x, y, and z are nonzero real numbers. Let

$$\begin{aligned} a &= 50x^2 + 9y^2 + 5z^2 + 6xy + 30xz + 6yz, \\ b &= 25x^2 + 41y^2 + 5z^2 + 40xy + 20xz + 26yz, \\ c &= 35x^2 + 15y^2 + 5z^2 + 33xy + 25xz + 16yz. \end{aligned}$$

Given that $c^2 = ab$, find all possible values of $\frac{x}{y}$.

SOLUTION. We can rewrite the desired equation as

$$[(7x + 2z)^2 + (x + 3y + z)^2][(5x + 4y + 2z)^2 + (5y + z)^2] \\ \geq [(7x + 2z)(5x + 4y + 2z) + (x + 3y + z)(5y + z)]^2.$$

By Cauchy-Schwarz, the only way we get equality is when

$$\begin{aligned} \frac{7x + 2z}{5x + 4y + 2z} &= \frac{x + 3y + z}{5y + z} \\ (7x + 2z)(5y + z) &= (5x + 4y + 2z)(x + 3y + z) \\ 0 &= 5x^2 - 16xy + 12y^2. \end{aligned}$$

Solving this quadratic, we get $\frac{x}{y} \in \left\{\frac{6}{5}, 2\right\}$.