Berkeley Math Circle: Monthly Contest 3 Solutions

1. Suppose you have 9 evenly spaced dots in a circle on a piece of paper. You want to draw a 9-pointed star by connecting dots around the circle without lifting your pencil, skipping the same number of dots each time.

   Determine the number of different stars that can be drawn, if the regular nonagon does not count as a star.

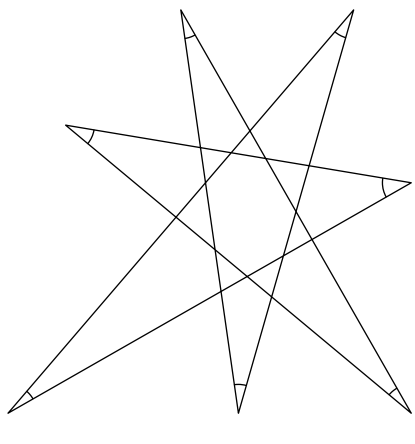
   SOLUTION.

   There are two different stars that can be drawn, by skipping 1 dot each time (as in the first star shown above) or skipping 3 dots each time (as in the third star shown above). Skipping 2 dots each time does not work because we end up drawing three separate equilateral triangles and having to lift our pencil, and skipping any other number of dots gives the regular nonagon or one of the three stars above.

2. Find (with proof) the units digit of the product of any 5 consecutive positive integers (*consecutive* means all in a row, like 5, 6, 7, 8, 9).

   SOLUTION. For any 5 numbers in a row, one of them must be a multiple of 5. Also, at least two of them are even. Thus, the product will be even and a multiple of 5, so it has units digit 0.

3. Consider the diagram on the right. What is the sum of the seven marked angles?

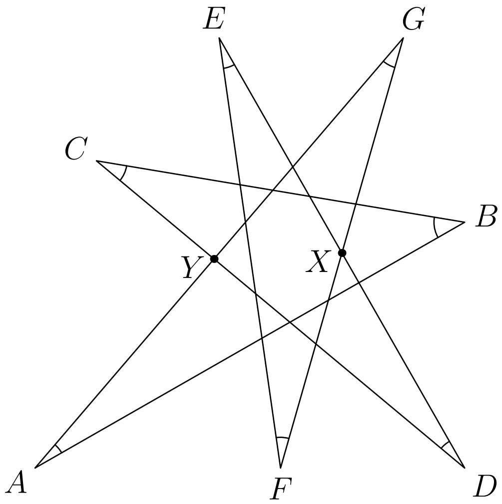


   SOLUTION.

Label the points as shown. We know that the sum of the angles in $\triangle FEX$, quadrilateral $GXDY$, and quadrilateral $CBAY$ are $180°, 360°$, and $360°$, respectively. Adding these, we get

$$\begin{aligned}
&180° + 360° + 360° \\
&= (\angle XFE + \angle FEX + \cancel{\angle EXF}) \\
&\quad + ((360° - \cancel{\angle DXG}) + \angle XDY + \cancel{\angle DYG} + \angle YGX) \\
&\quad + (\angle CBA + \angle BAY + (360° - \cancel{\angle CYA}) + \angle YCB) \\
&= \angle GFE + \angle FED + 360° + \angle EDC + \angle AGF \\
&\quad + \angle CBA + \angle BAG + 360° + \angle DCB.
\end{aligned}$$

Thus, these 7 angles add up to $\boxed{180°}$.



4. The 2020 members of the society of game theorists are holding the annual election for their leadership board. All members are initially on the board, and are ranked based on their qualifications. They start off by voting on whether to keep the board the same size; if they fail to get a strict majority, the member with the lowest rank is expelled. This process continues until they finally vote to keep the size of the board intact.

   It is common knowledge that, as each board member seeks to maximize their own influence, they seek to remain on the board while retaining as few other members as possible.

   At the end of this process, how many society members will remain on the leadership board?

   SOLUTION. Number members #1 to #2020, with #1 being the member with the highest rank. We claim that only if there are $2^n - 1$ members still on the board for some $n \in \mathbb{N}$, they all remain on the board.

   Proof. We use induction on n.

   Base case: $n = 1$.

   If there were $2^1 - 1 = 1$ persons on the board, that person would vote to keep themselves on the board and win.

   Inductive step: $n \implies n + 1$

   If there were at least 2^n members, the first $2^n - 1$ would all vote "no", because by induction they know they would be guaranteed a spot on the board in any case and want to minimize its total size. This would be a weak majority vote for "no" if the total size of the board were at most $2(2^n - 1) = 2^{n+1} - 2$ members.

   However, if there were $2^{n+1} - 1$ members, all whose numbers are at least 2^n would vote "yes", as by the the above, they would otherwise get kicked off the board. This is a strict majority, as desired.

   The largest number at most 2020 of the form $2^n - 1$ is 1023, so this is our answer. □

5. Let $P(x)$ be a polynomial with positive real coefficients. Prove that

$$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$$

holds for all positive real numbers x given that it holds for $x = 1$.

SOLUTION. Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

By Cauchy-Schwarz, we have

$$\begin{aligned} P(x)P\left(\frac{1}{x}\right) &= (a_n x^n + \cdots + a_1 x + a_0)\left(\frac{a_n}{x^n} + \cdots + \frac{a_1}{x} + a_0\right) \\ &\geq (a_n + \cdots + a_1 + a_0)^2 = P(1)^2 \geq 1. \end{aligned}$$

6. Let P be a polynomial with integer coefficients such that $P(2020) = P(2021) = 2021$. Prove that P has no integer roots.

SOLUTION. Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

If we plug in $x = 2020$, all terms except a_0 will be even, so since $P(2020)$ is odd, a_0 must be odd. But then if we plug in any other even number for x, $P(x)$ will still be odd since all the terms except a_0 will still be even, so $P(x)$ cannot be 0. Thus, $P(x)$ can have no even roots.

However, if we plug in an odd number for x, the term $a_i x^i$ in $P(x)$ has the same parity as the term $a_i \cdot 2021^i$ in $P(2021)$ for each $i = 1, \ldots, n$. Thus, the sum $P(x)$ has the same parity as the sum $P(2021)$. But $P(2021)$ is odd, so $P(x)$ must be odd as well, and so not 0. Thus, $P(x)$ cannot have odd roots either, so it has no integer roots.

7. Let

- P be a point inside a triangle $\triangle ABC$,
- $\triangle DEF$ be the pedal triangle of P, i.e., let D, E, F be the feet of the altitudes from P to BC, CA, AB, respectively,
- I be the incenter of $\triangle ABC$, and
- $\triangle XYZ$ be the Cevian triangle of I, i.e., X, Y, Z be the intersections of AI, BI, CI with BC, CA, AB, respectively.

Show that there is a triangle with side lengths PD, PE, and PF if and only if P is inside $\triangle XYZ$.

SOLUTION. While there is a synthetic solution, we present a solution using (unnormalized) barycentric coordinates.

Let $P = (x, y, z)$ in barycentric coordinates. We know that $PD = \frac{2[\triangle BCD]}{a} = 2[\triangle ABC]\frac{x}{a}$, so by SSS similarity if PD, PE, PF forms a triangle, so do $x/a, y/b, z/c$.

The equation $\frac{x}{a} = \frac{y}{b} + \frac{z}{c}$ is a line. However, we already know two points on the line, $Y = (a, 0, c)$ and $Z = (a, b, 0)$. Thus, $\frac{x}{a} < \frac{y}{b} + \frac{z}{c}$ is some side of line YZ. We know that this side does not contain $A = (1, 0, 0)$, so the region of $\triangle ABC$ on the side of the line must be $BCYZ$. Intersecting this with the other regions ($CAZX$ and $ABXY$) gives the desired.