Berkeley Math Circle: Monthly Contest 2 Solutions

1. Find the units digit of 17^{2021}.

   SOLUTION. The units digits of powers of 17 cycle: $7, 9, 3, 1, 7, 9, 3, 1, \ldots$, so the units digit of 17^n is 1 whenever n is a multiple of 4. Since 2020 is a multiple of 4, 17^{2020} has units digit 1, so 17^{2021} has units digit $\boxed{7}$.

2. Let $ABCD$ be a convex quadrilateral. Let I_A be the incenter of BCD and define I_B, I_C, I_D similarly. Show that $AC \perp I_BI_D$ if and only if $I_AI_C \perp BD$.

   SOLUTION. We show that both are true if and only if $AB + CD = BC + DA$.

   Let's first consider the triangle ABD. We know the incircle of ABD is centered at I_C. Let X, Y, T_A denote the points where the incircle is tangent to AB, AD, and BD, respectively. We have that $AX = AY, BX = BT_A$, and $DT_A = DY$ since each of these are segments tangent to the incircle from a common point. Thus,

   $$AB - AD = (AX + XB) - (AY + YD) = BX - CY = BT_A - DT_A.$$

   We now include the triangle BCD. Let T_C denote the point at which the incircle of BCD, centered at I_A, is tangent to BD. Replicating the argument for ABD, we see that $BT_C - DT_C = BC - CD$. Now, $I_AI_C \perp BD$ iff $T_A = T_C$, since the altitudes from I_A and I_C to diagonal BD must meet at a point. Hence, $I_AI_C \perp BD$ is equivalent to

   $$BA - DA = BT_A - DT_A = BT_C - DT_C = BC - CD,$$

   which in turn is equivalent to $AB + CD = AD + BC$.

   We use a symmetric argument, but this time considering triangles ABC and ABD, to show that a necessary and sufficient condition for $I_BI_D \perp AC$ is also $AB + CD = AD + BC$. Since $AC \perp I_BI_D$ and $I_AI_C \perp BD$ are both equivalent to $AB + CD = AD + BC$, we have $AC \perp I_BI_D \iff I_AI_C \perp BD$.

3. Cheryl chooses a word in this problem and tells its first letter to Aerith and its last letter to Bob.[1][2] The following conversation ensues over a series of emails:

   Aerith: "I don't know her word, do you?"
   Bob: "No, in fact, I don't know if we can ever figure out what her word is without having more information."
   Aerith: "Then I do know what it is!"
   Bob: "Now I also do."

[1] Aerith and Bob are both aware of this setup.
[2] Footnotes are not considered to be part of the problem statement.

What is Cheryl's word?

SOLUTION. We will process their conversation message by message.

<u>"I don't know her word, do you?"</u>

Aerith would know Cheryl's word if the first letter was unique, so Aerith does not have any of b (Bob), k (know), m (more), p (problem), s (series), y (you).

<u>"No, in fact, I don't know if we can ever figure out what her word is without having more information."</u>

If they can't ever figure out what Cheryl's word is, even if they just outright told each other their letters, it would be because Cheryl's word is not identifiable from these letters alone. The unidentifiable words are: can/conversation, Cheryl's/chooses, emails/ensues, fact/first, in/information, is/its, tells/this, what/without.

Bob's statement is then equivalent to the claim "The word could be in this list but does not have to be." The only last letters which satisfy this property are n and t. While some words in the list end in s, s does not satisfy the property because no words not in the list end in s. (Series was the only one but it was eliminated by Aerith's statement.)

<u>"Then I do know what it is!"</u>

In order for Aerith to make this claim, there must be only one possible word beginning with her letter that ends with n or t. Her letter must then be one of d (don't), l (last), o (out), t (then).

<u>"Now I also do."</u>

For Bob to know Cheryl's word, the last letter must be unique among the remaining options. Inspecting we see that the word must then be "then".

4. Find all polynomials f that satisfy the equation $\frac{f(9x)}{f(3x)} = \frac{243x-729}{x-81}$ for infinitely many values of x.

SOLUTION. We have

$$(x-81)f(9x) = (243x-729)f(3x)$$

for infinitely many values of x. Since both sides of this equation are polynomials, they must then be equal for all x.

Plugging in $x = 3$, we get $f(27) = 0$. Plugging in $x = 9$ then gives $f(81) = 0$. Plugging in $x = 27$ then gives $f(243) = 0$. Thus we have that $f(x) = g(x) \cdot (x-27)(x-81)(x-243)$ for some polynomial $g(x)$.

Plugging this into the original equation, we get that for infinitely many x,

$$243\frac{x-3}{x-81} = \frac{f(9x)}{f(3x)} = \frac{g(9x)}{g(3x)} \cdot \frac{(9x-27)(9x-81)(9x-243)}{(3x-27)(3x-81)(3x-243)} = \frac{g(9x)}{g(3x)} \cdot 27\frac{x-3}{x-81},$$

so $\frac{g(9x)}{g(3x)} = \frac{243}{27} = 9$ and thus f works if and only if $g(9x) = 9g(3x)$ for all x. Thus, the roots of $g(9x)$ are those of $g(3x)$, so they must all be zero, so g must be of the form ax^n. Plugging in $x = 1$ then gives $9^n = 9 \cdot 3^n$, so $n = 2$. Indeed, $g(x) = x^2$ works.

The solutions are then

$$f(x) = ax^2(x-27)(x-81)(x-243)$$

for real constants a.

5. Without a calculator, find a factor $85^9 - 21^9 + 6^9$ that is between 2000 and 3000.

SOLUTION. We know that $85^9 - 21^9$ has $85 - 21 = 64$ as a factor, and 6^9 also has 64 as a factor, so the sum is divisible by 64.

Similarly, $-21^9 + 6^9$ is divisible by $-21 + 6 = -15$, which means it is divisible by 5. Since 85^9 is also divisible by 5, the whole sum is divisible by 5.

Finally, $85^9 + 6^9$ is divisible by $85 + 6 = 91$, so it is divisible by 7. Since 21^9 is also divisible by 7, the sum is divisible by 7.

Since the sum is divisible by 64, 5, and 7, it is also divisible by $64 \cdot 5 \cdot 7 = \boxed{2240}$.

6. For which $n \geq 3$ can an $n \times n$ board with all four corners removed be tiled with "L" and "J" tetris pieces?

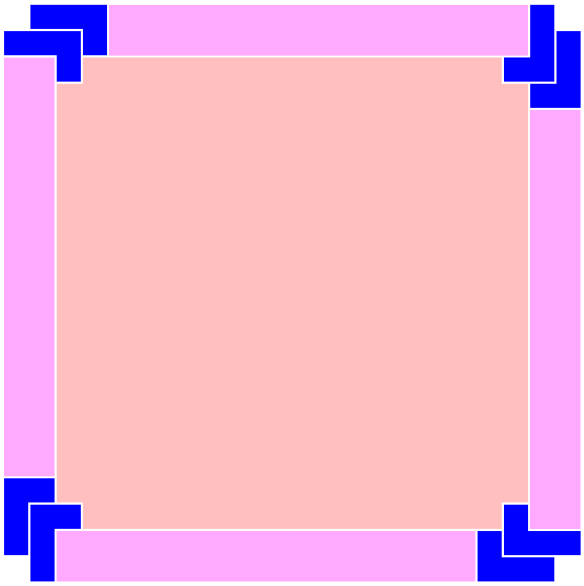
SOLUTION. We claim that n works iff $n \equiv 2 \pmod 4$.

The board has $n^2 - 4$ squares. If n is odd, this is odd, so the board wouldn't be coverable with tetris tiles (which have 4 cells each).

Thus n is even. Alternately color the rows of the board black and white. Each the tetris pieces must cover three squares of one color and one of the other. Let b be the number of blocks which cover three black squares and w be the number of blocks which cover three white squares.

By vertical symmetry, there are an equal amount of black squares and white squares, so $b = w$. The total number of tiles is thus $b + w = 2b$, so the total number of cells is $8b = n^2 - 4$. Thus, $8|n^2 - 4$, so $8 \nmid n^2$, so $4 \nmid n$. Thus, $n \equiv 2 \pmod 4$, as desired.

It remains to show that all n such that $n \equiv 2 \pmod 4$ work. In fact, one only needs "J" tiles to construct such n; We present the following proof by induction:



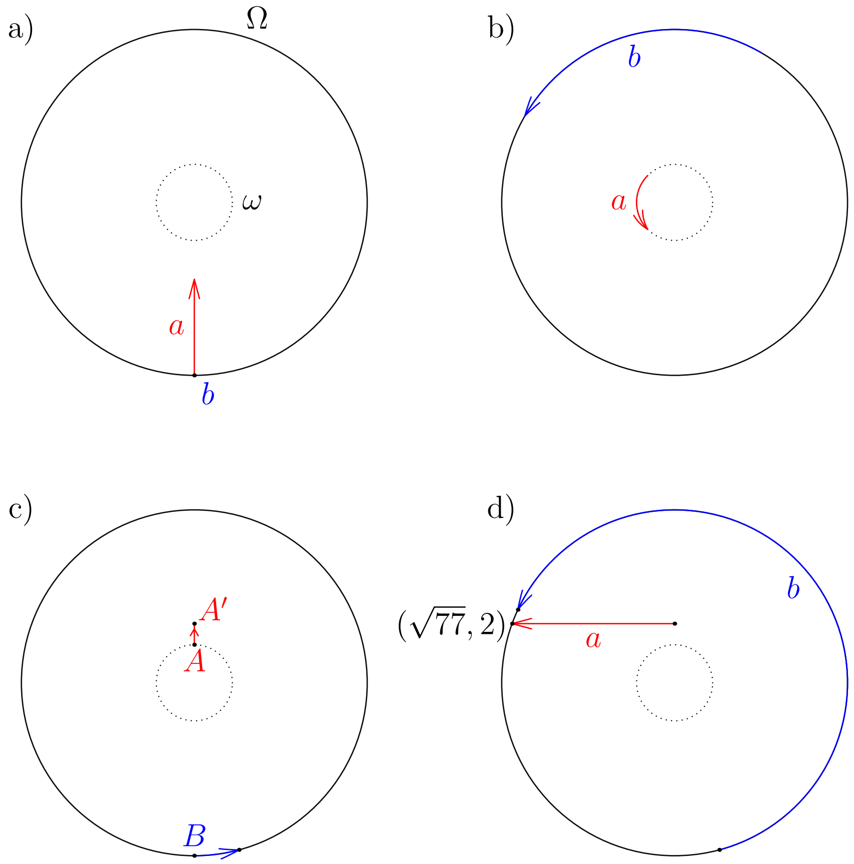
Our base case is $n = 2$, which loses its tiles once we remove the corners, so it is trivially tiled by 0 pieces. For $n \equiv 2 \pmod 4$, we can follow the process illustrated in the illustration: The pink rectangles have length $n - 6 \equiv 0 \pmod 4$ and width 2, so we can nest J blocks to form a rectangle of width 2 and length 4, which we stack to tile the pink regions. The red region is a board with length $n - 4 \equiv 2 \pmod 4$ and its corners removed. By induction, such a board can be tiled by our blocks, so we're done.

7. Aerith and Bob are playing tag at Lake Round, a perfectly circular lake. Aerith tags Bob right next to the lake and dives in. Aerith can swim at a speed of 2mph, while Bob can't swim but runs at a speed of 9mph. Can Aerith leave the lake without getting tagged?[3]

SOLUTION. We claim that Aerith can escape, even when Bob plays optimally and always runs towards Aerith's current location.

Let a dash (d) be a unit of distance and a tick (t) be a unit of time such that the radius of the lake is 9 dashes and such that one mph is equal to one dash per tick (d/t). Let Ω be the perimeter of the lake, and let ω be a concentric circle with radius 1.98 dashes.

[3]Assume that they always know each other's locations and can instantaneously accelerate.



We outline a four-step strategy for Aerith to escape the lake:

a) First, Aerith swims to the edge of ω.

b) Then, Aerith begins swimming along ω. Her angular velocity is then

$$\left(2\frac{\mathrm{d}}{\mathrm{t}}\right)\Big/(1.98\mathrm{d}) > 1\mathrm{t}^{-1}.$$

However, Bob's angular velocity is exactly $\left(9\frac{\mathrm{d}}{\mathrm{t}}\right)/(9\mathrm{d}) = 1\mathrm{t}^{-1}$, which is slightly less, so Aerith outpaces Bob. Thus after some amount of time, Aerith will eventually be at the furthest point on ω from Bob. At this point in time, let Aerith's current position be A and let Bob's current position be B.

c) Aerith now swims outward on line AB until she reaches the point A' that is 2d away from the center of the lake. This amount of time this takes is

$$(2\mathrm{d} - 1.98\mathrm{d})\Big/\left(2\frac{\mathrm{d}}{\mathrm{t}}\right) = 0.01\mathrm{t},$$

and in this time Bob travels $0.01\mathrm{t} \cdot 9\frac{\mathrm{d}}{\mathrm{t}} = 0.09\mathrm{d}$.

d) Finally, Aerith swims away from Bob along the straight line through A' perpendicular to line AB. Aerith's angular velocity along this journey is now less than $1\mathrm{t}^{-1}$ because, as we saw earlier, she has to be within a circle of radius 2 in order to outpace Bob. Therefore, she remains ahead of Bob, so Bob will continue in his direction when chasing her. It takes her $\left(\sqrt{(9\mathrm{d})^2 - (2\mathrm{d})^2}\right)\Big/\left(2\frac{\mathrm{d}}{\mathrm{t}}\right) = \frac{\sqrt{77}}{2}\mathrm{t}$ to reach the edge, and in this time Bob can travel $\frac{\sqrt{77}}{2}\mathrm{t} \cdot 9\frac{\mathrm{d}}{\mathrm{t}} = \frac{9\sqrt{77}}{2}\mathrm{d}$.

The total distance Bob can travel along the circle in the time it takes Aerith to reach the edge is thus $0.09\mathrm{d}+\frac{9\sqrt{77}}{2}\mathrm{d} < 40\mathrm{d}$. However, in order to arrive on time to tag Aerith, he would need to travel $9\left(\frac{3\pi}{2} - \arcsin\left(\frac{2}{9}\right)\right) > 40\mathrm{d}$, so Aerith will indeed escape before Bob catches up.