Berkeley Math Circle: Monthly Contest 1 Solutions

1. Order the numbers $2^{300}, 10^{100}$, and 3^{200} from least to greatest, and prove that your ordering is correct.

   SOLUTION. We note that $2^{300} = (2^3)^{100} = 8^{100}$ and $3^{200} = (3^2)^{100} = 9^{100}$. Since $8^{100} < 9^{100} < 10^{100}$, the ordering is $\boxed{2^{300} < 3^{200} < 10^{100}}$.

2. Determine the number of convex polygons all of whose sides are the square roots of positive integers which can be inscribed in a unit circle. Polygons that are rotations or reflections of each other are considered the same.

   SOLUTION. Any chord in a unit circle has length at most 2 (if it is a diameter), so all side lengths must be $1, \sqrt{2}, \sqrt{3}$, or 2. We can do casework based on the number of sides:

   - *Triangles:* The possible combinations of side lengths are $(\sqrt{2}, \sqrt{2}, 2)$, $(1, 1, \sqrt{3})$, $(1, \sqrt{3}, 2)$, and $(\sqrt{3}, \sqrt{3}, \sqrt{3})$. This is 4 possibilities.
   - *Quadrilaterals:* The possible ordered combinations of side lengths are $(1, 1, 1, 2)$, $(1, \sqrt{3}, \sqrt{2}, \sqrt{2})$, $(1, \sqrt{3}, 1, \sqrt{3})$, $(1, \sqrt{3}, \sqrt{3}, 1)$, $(\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2})$, and $(1, \sqrt{2}, \sqrt{3}, \sqrt{2})$. This is 6 possibilities.
   - *Pentagons:* The possible combinations of side lengths are $(1, 1, 1, 1, 1, \sqrt{3})$, $(1, 1, 1, \sqrt{2}, \sqrt{2})$, and $(1, 1, \sqrt{2}, 1, \sqrt{2})$. This is 3 possibilities.
   - *Hexagons:* The only option is that all sides are 1, so there is 1 possibility.

   Putting these together, there are $4 + 6 + 3 + 1 = \boxed{14}$ polygons total.

3. Aerith and Bob play a game where they start with a row of 20 squares, and take turns crossing out one of the squares. The game ends when there are two squares left. Aerith wins if the two remaining squares are next to each other, and Bob wins if they are not next to each other.

   a) If Aerith goes first, who has a winning strategy?

   b) What if Bob goes first?

   SOLUTION.

   a) Bob has a winning strategy. Since he goes second, he will get to take the last turn. If none of the last three squares remaining are together, he wins no matter what. If they are all in a row, he chooses the middle one, and if they are two together and one separate, he chooses one of the two together.

   b) Aerith has a winning strategy. Number the squares 1 to 20. Whenever Bob chooses square $2k - 1$, Aerith chooses square $2k$, and whenever Bob chooses square $2k$, Aerith chooses square $2k - 1$. This guarantees that the last two remaining squares will be $2k$ and $2k - 1$ for some k, so they will be together.

4. For positive integers a, b, c, x, y, z such that $axy = byz = czx$, can $a+b+c+x+y+z$ be prime?

SOLUTION. Dividing by xyz, we get $a/x = b/y = c/z$. Let this fraction in lowest terms be m/n. We then have

$$m + n \mid a + x, b + y, c + z,$$

so $m + n$ is a nontrivial factor of $a + b + c + x + y + z$, so $a + b + c + x + y + z$ is not prime.

5. Aerith rolls a fair die until she gets a roll that is greater than or equal to her previous roll. Find the expected number of times she will roll the die before stopping.

SOLUTION. The number of rolls is always at least 2 and at most 7. For there to be at least k rolls, the first $k-1$ rolls have to be distinct and in decreasing order. The number of ways this can happen is $\binom{6}{k-1}$, and the total number of ways to have $k-1$ rolls is 6^{k-1}. Thus, the expected value is

$$\begin{aligned} E(\#\text{ of rolls}) &= P(\geq 1\text{ roll}) + P(\geq 2\text{ rolls}) + \cdots + P(\geq 7\text{ rolls}) \\ &= 1 + \frac{\binom{6}{1}}{6} + \frac{\binom{6}{2}}{6^2} + \frac{\binom{6}{3}}{6^3} + \frac{\binom{6}{4}}{6^4} + \frac{\binom{6}{5}}{6^5} + \frac{\binom{6}{6}}{6^6}. \end{aligned}$$

By the binomial theorem, this is $\boxed{\left(1 + \frac{1}{6}\right)^6 \approx 2.52}$.

6. Find all positive integers N, n such that N^2 is 1 away from $n(N+n)$.

SOLUTION. The solutions are $(N, n) = (F_{i+1}, F_i)$.

If $N > n$, $N^2 - n(N+n) = N(N-n) - n^2$, so (N, n) works if and only if $(n, N-n)$ works.

If $N \leq n$, $n(N+n) - N^2 \geq n^2 \geq 1$, so the only solution is $(N, n) = (1, 1)$.

Thus, all solutions eventually become $(1, 1)$ after repeadtely applying the operation $(N, n) \mapsto (n, N-n)$, so reversing this process, every solution comes from repeatedly applying the operation $(N, n) \mapsto (N+n, N)$ to $(1, 1)$.

This operation maps (F_i, F_{i-1}) to (F_{i+1}, F_i), and $(1, 1) = (F_2, F_1)$, giving the desired via induction.

7. In $\triangle ABC$, suppose the incircle has center I and is tangent to BC at D, and the A-excircle has center I_a and is tangent to BC at D'. Show that ID' and I_aD intersect on the altitude from A to BC.

SOLUTION. The intersection point is the midpoint of the altitude.

Let E be the point on the incircle diametrically opposite from D. Then the homothety centered at point A which takes the incircle to the A-excircle takes E to D', so A, D', and E are collinear. Since DE is parallel to the altitude of $\triangle ABC$ and AI bisects DE, it also bisects the altitude.

Similarly, let E' be the point on the A-excircle diametrically opposite from D'. Then the same homothety as before takes D to E', so A, D, and E' are collinear. Since $D'E'$ is parallel to the altitude of $\triangle ABC$ and DI_a bisects $D'E'$, it bisects the altitude as well. Thus, both lines pass through the midpoint of the altitude.