Berkeley Math Circle: Monthly Contest 8 Solutions

1. Suppose you have an unlimited number pennies, nickels, dimes, and quarters. Determine the number of ways to make 30 cents using these coins.

   Solution. We use cases to organize our work, based first on the number of quarters and then the number of dimes. First note that the number of quarters must be 0 or 1, since 2 quarters would be too much. This gives 2 cases:

   - **1 quarter:** There are 2 possibilities: a quarter and a nickel or a quarter and 5 pennies.
   - **0 quarters:** If we don't use quarters, we can use at most 2 dimes, so we can make 3 subcases based on the number of dimes:
     - **2 dimes:** We need to make 10 cents more using nickels or pennies. We could use 0, 1, or 2 nickels, so there are 3 possibilities.
     - **1 dime:** We need to make 20 cents more using nickels or pennies. We could use 0, 1, 2, 3, or 4 nickels, so there are 5 possibilities.
     - **0 dimes:** We need to make 30 cents more using nickels and pennies. We could use 0, 1, 2, 3, 4, 5, or 6 nickels, so there are 7 possibilities.

   Putting this together, we get a total of $2 + 3 + 5 + 7 = 17$ possibilities. □

2. In a certain two-player game, you start with a rectangular $m \times n$ grid of squares. On each turn, a player either makes a horizontal cut and takes away the portion of the rectangle above the cut, or makes a vertical cut and takes away the portion to the right. Whichever player takes the last square (in the bottom left corner) loses. If both players play perfectly, determine for which values of m and n the first player will win.

   Solution. The first player wins whenever the grid is not a square (i.e. $m \neq n$). The strategy is to always leave the other player with a square grid (where $m = n$), since then they will eventually be left with a 1×1 square and will lose. This works since starting from a square, it is impossible to get to a smaller square, but starting from a non-square, it is always possible to get to a square. Thus, the first player wins whenever the starting grid is not a square, and the second player wins whenever the starting grid is a square. □

3. A circle is inscribed in $\triangle ABC$ that touches side BC at D, side AC at E, and side AB at F. Show that $\triangle DEF$ must be acute.

   Solution. (Problem source: Aops) Since AE and AF are tangents from the same point to the same circle, we must have $AE = AF$. Thus, $\triangle AEF$ is isosceles, so

   $$\angle AEF = \angle AFE = \frac{180^\circ - \angle A}{2} = 90^\circ - \frac{1}{2}\angle A.$$

   Now, since AE is a tangent to the circle, $\angle EDF = \angle AEF = 90^\circ - \frac{1}{2}\angle A$. Thus, $\angle EDF < 90^\circ$. Similarly, the other two angles must also be less than 90°, so the triangle is acute. □

4. Prove that every positive real number y satisfies

$$2y \geq 3 - \frac{1}{y^2}.$$

When does equality occur?

Solution. (Problem source: Aops) Since y is positive, we can multiply both sides by y and rearrange to get the equivalent inequality

$$2y^3 - 3y^2 + 1 \geq 0.$$

This factors as

$$(2y + 1)(y - 1)^2 \geq 0.$$

The first factor is positive since $y > 0$, and the second is nonnegative since it is a real number squared. Thus, the product must be nonnegative, and will be 0 iff $(y - 1)^2 = 0$, which happens iff $y = 1$.

Alternate solution: By AM-GM, we get

$$\frac{\frac{1}{y^2} + y + y}{3} \geq \sqrt[3]{\frac{1}{y^2} \cdot y \cdot y} = 1.$$

Multiplying by 3 and rearranging gives the desired inequality, and equality occurs iff $\frac{1}{y^2} = y$, which is true iff $y = 1$. □

5. It's a week before Thanksgiving, and a family is trying to find their turkey. There are 5 boxes in a row, and the turkey is hiding in one of the 5 boxes.

   Every day, the family is allowed to check one box to try to find the turkey, and every night, the turkey moves to a box right next to the box it was in. For example, from box 3 it could move to box 2 or 4, and from box 5 it must move to box 4. Determine a strategy for the family to catch the turkey before Thanksgiving.

Solution. One strategy is as follows:

First assume the turkey starts in an even numbered box (box 2 or 4). On day 1, check box 2. If the turkey is not there, he must have been in box 4 to begin with, so tomorrow he will be in box 3 or box 5. On day 2, check box 3. If the turkey is not there, then he must currently be in box 5 (since he started in box 4), so tomorrow he will be in box 4. Thus, on day 3, check box 4.

If he is not there, then the assumption that he began in an even numbered box must have been wrong. Thus, he started in an odd numbered box, which means on day 4 he will be in an even numbered box. Thus, we can repeat the same strategy of checking boxes 2, 3, and 4 on days 4, 5, and 6, and we are guaranteed to find the turkey in at most 6 days. □

6. A piece of origami paper is green on one side and white on the other. The paper has been folded along four or more straight line segments, all meeting at the same vertex, such that the folded model lies flat. A fold is a **mountain fold** if the white sides of the paper are touching and the green is on the outside, and a **valley fold** if the green sides are touching and the white is on the outside.

a) The line segments meeting at the vertex form a number of angles, one between each adjacent pair of segments. Suppose we color half the angles white and half the angles black such that no two angles next to each other are the same color. Show that the sum of the black angles is $180°$.

b) Show that the number of mountain folds and the number of valley folds meeting at the vertex differ by exactly 2.

Solution. a) Suppose we draw a circle on the paper centered at the vertex. Now, suppose the paper is folded up, so arcs of the circle are layered on top of each other. We can start walking along this circle, and every time we reach an edge (i.e. change from a white part to a black part or vice versa) we turn around and start walking in the other direction. Thus, for white portions we are always walking along the circle in one direction, and for black portions we are walking in the other direction. Since the walk ends at the same edge we started at, the sum of the black arcs we walked along must equal the sum of the white arcs. Thus, the sum of the black angles equals the sum of the white angles, so both sums are $180°$.

b) Write M for a mountain fold and V for a valley fold. Not all the folds can be valleys or all mountains, since then the paper could not lie flat. Thus, if there are four or more folds, there must be three adjacent folds in the sequence MVM, VMV, MMV, or VVM. In each case we can "squash fold" to turn the three folds into a single fold such that the model still lies flat, and the resulting fold will be an M, V, M, and V, respectively.

Thus, at each step, the difference between the number of M's and the number of V's does not change. We can keep doing this until there are only two folds left. These two folds must then actually be in the same straight line, and must both be mountains (in which case there are two more mountains) or both valleys (in which case there are two more valleys). In either case, the numbers of mountain and valley folds differ by 2.

□

7. In the pattern shown below, row 1 (the bottom row) consists of two 1's, and row n is formed by taking row $n-1$ and inserting between each adjacent pair of numbers a and b their sum $a+b$:

1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
1		4		3		5		2		5		3		4		1
1				3				2				3				1
1								2								1
1																1

In row 2019 of the pattern, how many copies of 2019 will there be?

Solution. The number of copies of n in row n is $\phi(n)$, where ϕ is the totient function, or the number of positive integers less than or equal to n that are relatively prime to n. To see this, consider the following pattern of fractions, where we start with $\frac{0}{1}$

and $\frac{1}{1}$ and insert $\frac{a+c}{b+d}$ between $\frac{a}{b}$ and $\frac{c}{d}$:

$$\begin{array}{ccccccccccccccccc}
\frac{0}{1} & \frac{1}{5} & \frac{1}{4} & \frac{2}{7} & \frac{1}{3} & \frac{3}{8} & \frac{2}{5} & \frac{3}{7} & \frac{1}{2} & \frac{4}{7} & \frac{3}{5} & \frac{5}{8} & \frac{2}{3} & \frac{5}{7} & \frac{3}{4} & \frac{4}{5} & \frac{1}{2} \\
\frac{0}{1} & & \frac{1}{4} & & \frac{1}{3} & & \frac{2}{5} & & \frac{1}{2} & & \frac{3}{5} & & \frac{2}{3} & & \frac{3}{4} & & \frac{1}{1} \\
\frac{0}{1} & & & & \frac{1}{3} & & & & \frac{1}{2} & & & & \frac{2}{3} & & & & \frac{1}{1} \\
\frac{0}{1} & & & & & & & & \frac{1}{2} & & & & & & & & \frac{1}{1} \\
\frac{0}{1} & & & & & & & & & & & & & & & & \frac{1}{2}.
\end{array}$$

The denominators of these fractions form the pattern of numbers in the problem, and the fractions are in increasing order. Row n contains all the fractions in the nth Farey sequence exactly once, i.e. all the fractions with denominator at most n (plus some other fractions). Thus, the number of copies of n in row n equals the number of fractions between 0 and 1 with denominator n in simplest form. Each of these fractions must have numerator less than or equal to n and relatively prime to n, so there are $\phi(n)$ such fractions. Thus, there are $\phi(n)$ copies of n in row n.

The prime factorization of 2019 is $3 \cdot 673$, thus in row 2019 the number of copies of 2019 is

$$\phi(2019) = 2019\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{673}\right) = 2 \cdot 672 = 1244.$$

□