Berkeley Math Circle: Monthly Contest 7 Solutions

1. The numbers 1 to 10 are written on a blackboard in a row. Determine whether it is possible to put nine + and − signs between them (one between each pair of adjacent numbers) to make the resulting sum 0.

   Solution. It is not possible. Note that the sum of the numbers is

   $$1 + 2 + 3 + \cdots + 9 + 10 = 55,$$

   an odd number. Thus, if we put all + signs, the sum is odd. Every time we change a + to a −, we change a $+n$ term to a $-n$ term for some n, which subtracts $2n$ from the sum. This changes the sum by an even number, so the sum will still be odd. Thus, no matter what, the sum is odd and therefore cannot be 0. □

2. In a certain kingdom, the only coin values are 3 and 5. Determine all possible amounts of money you can have using only these coins.

   Solution. The amounts of money that you can have are all positive integers except 1, 2, 4, and 7. We can see that all these are impossible. To show that all other amounts are possible, it is clear that 3 and 5 are possible. Also, 6, 8, 9, and 10 are all possible, since

   $$\begin{aligned} 6 &= 3 + 3, \\ 8 &= 3 + 5, \\ 9 &= 3 + 3 + 3, \\ 10 &= 5 + 5. \end{aligned}$$

   Now, any higher amount of amount of money has a remainder of 0, 1, or 2 when divided by 3. If it has a remainder of 0, it is 9 plus a multiple of 3, if it has a remainder of 1, it is 10 plus a multiple of 3, and if it has a remainder of 2, it is 8 plus a multiple of 3. No matter what, it is one of 8, 9, or 10 plus a multiple of 3, so we can just add more 3 coins to get from one of 8, 9, or 10 to our number. Thus, all higher amounts of money can be made. □

3. How many ways are there to list the numbers 1 to 10 in some order such that every number is either greater or smaller than all the numbers before it?

   Solution. (Problem source: Aops) The number of lists is $2^9 = 512$. Imagine building a list from the end. First, note that the last number must be 1 or 9, since otherwise it would be less than some numbers before it and greater than others. Thus, there are two choices for the last number.

   Similarly, after choosing the last number, the second to last number must be either the biggest number remaining or the smallest number remaining, since otherwise it would be less than some numbers before it and greater than some other numbers before it. Thus, we have two choices for the second to last number. similarly, there are two choices for the 3rd number from the end once the last two are chosen, and so on until we reach the second number in the list. For the first number in the list, there is only one choice since there is only one number remaining. Putting this together, we've had 2 choices for each of the last 9 spots and 1 choice for the first, giving $2^9 = 512$ possible lists. $\square$

4. A large integer is divisible by all the integers between 1 and 30 inclusive, except for two consecutive integers. Determine those two consecutive integers.

   Solution. The two numbers must be 16 and 17.

   First, note that if a number n is not a power of a prime, then it has two smaller factors a and b that are relatively prime and multiply to n. Thus, n is the least common multiple of a and b. Then, any multiple of both a and b must also be a multiple of n, so any multiple of all the numbers less than n would have to be a multiple of n (since two of the numbers less than n are a and b). Thus, for there to be some number that is a multiple of all the numbers less than n but not n itself, n must be a prime power. Therefore, both consecutive numbers are prime powers.

   Note also that if n is at most 15, then $2n \leq 30$. Thus, if we chose one of the consecutive numbers to be $n \leq 15$, the big number would have to be a multiple of $2n$, and therefore also of n. Thus, this cannot happen, so both numbers must be more than 15. The only pair of consecutive numbers from 16 to 30 that are both prime powers is 16 and 17, so this must be the answer. $\square$

5. Prove that every positive real number x satisfies

$$\sqrt{x^2 - x + \frac{1}{2}} \geq \frac{1}{x + \frac{1}{x}}.$$

   Solution. (Problem source: Aops) By the QM-AM inequality,

$$\sqrt{x^2 - x + \frac{1}{2}} = \sqrt{\frac{x^2 + (1-x)^2}{2}} \geq \frac{x + (1-x)}{2} = \frac{1}{2}.$$

   Now, by AM-GM (since we know x is positive),

$$\frac{x + \frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}} \implies \frac{1}{2} \geq \frac{1}{x + \frac{1}{x}}.$$

   Putting this together gives

$$\sqrt{x^2 - x + \frac{1}{2}} \geq \frac{1}{2} \geq \frac{1}{x + \frac{1}{x}}.$$

$\square$

6. On a certain island, there are knights, who always tell the truth, knaves, who always lie, and spies, who could do either. Suppose you meet three people, and you know one is a knight, one is a knave, and one is a spy, but you don't know which is which.

Find a method to ask three yes/no questions, each to one of the three people, so you can determine for sure which is which. You may question the same person multiple times, and your questions can depend on answers to previous questions.

Solution. One strategy is as follows:

1. Ask the first person, "Is the second person more likely to tell the truth than the third person?"

   If the response is "yes", then if the first person is the knight, the second must be the spy and the third must be the knave. If the first person is the knave, the second must be the spy and the third the knight. Either way, if the first person is not the spy, the second person is the spy.

   If the response is "no", then if the first person is the knight, the second must be the knave and the third the spy. If the first is the knave, the second must be the knight and the third the spy. Either way, the first person or the third is the spy.

   Thus, at this point, you either know for certain that the second person is not the spy, or that the third person is not the spy.

2. Ask the person whom you now know to be either the knight or the knave and not the spy (either the second or the third person) some question you know the answer to, like, "Does $1 + 1$ equal 2?" Then, depending on their response, you know whether they are a knight or a knave.

3. Now ask this same person about the identity of one of the other two people. Since you know for sure whether this person is lying, their answer will tell you the identities of the other two people. For instance, if you know the second person to be a knave, you could ask, "Is the first person a knight?" If the second person says yes, the first person is the spy and the third the knight, and if the second person says no, the first person is the knight and the third the spy. No matter what, you can now figure out the identities of all three people.

□

7. Given ten points in the plane, show that it is always possible to cover all of them with non-overlapping unit circles.

Solution. Suppose we place the circles in a hexagonal tiling so they are all tangent to each other. Then each circle has area π, and each circle is inscribed in a hexagon. In each hexagon, the distance from the center to each side is 1. Thus, each of the six equilateral triangles making up the hexagon has height 1 and thus side length $2/\sqrt{3}$. Therefore, the triangle's area is

$$\frac{1}{2} \cdot 1 \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Since the hexagon is made of six triangles, it has area $6 \cdot 1/\sqrt{3} = 2\sqrt{3}$. Thus, the portion of the plane covered by circles is the area of each circle divided by the area of the hexagon it is inscribed in, or

$$\frac{\pi}{2\sqrt{3}} \approx 0.907.$$

So, if we use unit circles in a hexagonal tiling with a randomly chosen center, each point has a 90.7% chance of being covered by a circle, so the expected number of our 10 points that are covered is $9.07 > 9$. Since every tiling covers a whole number of points and the expected number covered is more than 9, there must be some tiling that covers at least 10 points. $\square$