Berkeley Math Circle: Monthly Contest 6 Solutions

1. At a party with 100 people, everyone is either a knight, who always tells the truth, or a knave, who always lies. Each person says they shook hands with a different number of knights at the party, from 0 to 99. Each pair of people shook hands at most once, and everyone knows whether each other person is a knight or knave. Determine how many knights were at the party.

   Solution. Call the person who said they shook hands with i people person i for each i from 0 to 99. Now, if person 99 is telling the truth, person 99 must have shaken hands with everyone else, and all the other people must be knights. But then person 0 would be lying, since they said they shook hands with no knights but must have shaken hands with person 99. This is a contradiction, so person 99 must be a knave.

   Similarly, if person 98 is a knight, person 98 must have shaken hands with people 0 to 97, and they must all be knights. This again means person 0 would be lying, a contradiction, so person 98 is also a knave. Continuing in this manner, we find that person 97 is a knave, as is person 96, and so on down to person 1. Thus, only person 0 is telling the truth, so there were 99 knaves and one knight at the party. □

2. A castle has infinitely many rooms labeled $1, 2, 3, \ldots$, which are divided into several halls. Suppose room n is on the same hall as rooms $3n + 1$ and $n + 10$ for every n. Determine the maximum possible number of different halls in the castle.

   Solution. There are at most three different halls in the castle. Because rooms n and $n + 10$ are on the same hall, any two rooms with the same units digit must be on the same hall.

   Now, repeatedly using the rule that rooms n and $3n+1$ are on the same hall, we find that room 1 is on the same hall as rooms 4, 13, and 40, and 121, so all rooms with units digit 0, 1, 3, or 4 are on the same hall. There need not be any other rooms on this hall, since whenever n has units digit in this set, so does $3n + 1$, and so does $(n - 1)/3$, if it is a whole number.

   Similarly, room 2 is on the same hall as rooms 7 and 22, so all rooms with units digit 2 or 7 are on the same hall, and there need not be any other rooms on this hall. Finally, rooms 5, 16, 49, and 148 are on the same hall, so all rooms with units digit 5, 6, 8 or 9 are on the same hall. Thus, we see that there are at most three different halls in the castle. □

3. Two points A and C are marked on a circle; assume the tangents to the circle at A and C meet at P. Let B be another point on the circle, ans suppose PB meets the circle again at D. Show that $AB \cdot CD = BC \cdot DA$.

   Solution. (Problem source: Aops) Since PA is tangent to the circle and $\angle ABD$ is the inscribed angle opposite AD, $\angle PAD = \angle ABP$. Since $\angle APB$ is shared, it follows from AA similarity that $\triangle PAD \sim \triangle PBA$. Thus we get

$$\frac{AB}{AD} = \frac{PB}{PA}.$$

Similarly, $\triangle PCD \sim \triangle PBC$, so

$$\frac{BC}{CD} = \frac{PB}{PC}.$$

Since PA and PC are tangents from the same point to the same circle, they are the same length. Thus,

$$\frac{AB}{AD} = \frac{BC}{CD} \implies AB \cdot CD = BC \cdot DA.$$

□

4. Suppose you have three children and 40 pieces of candy. How many ways are there to distribute the candy such that each child gets more than one but fewer than 20 pieces?

   Solution. We can use the "stars and bars" method, since this is the equivalent of giving 40 pieces of candy to three children, such that each child gets at least two pieces. This is the same as giving $40 - 6 = 34$ pieces to the three children with no restrictions (since we can pretend six pieces are already given out, two to each child). This is the same as arranging the 36 pieces of candy and two "dividers" (one between the first child's candy and the second child's candy and one between the second child's candy and the third child's candy) in a row. Thus, the number of ways to do this is

$$\binom{36}{2} = \frac{36 \cdot 35}{2} = 630.$$

   However, this counts cases where one child gets at least 20 pieces of candy, which we don't want, so we need to subtract these cases. Only one child could get at least 20 pieces, or we would run out of candy, so there are three ways to choose the child who gets at least 20 pieces. Then, we have 20 pieces of candy to distribute among three children so that two of them (the ones who didn't already get 20) get at least 2 pieces each, which is the equivalent of distributing $20 - 4 = 16$ pieces among 3 children with no restrictions (since we can think of four of the pieces as being already given out). This can be done in $\binom{18}{2}$ ways, since it is the same as lining up 6 pieces of candy and two "dividers." Putting this together, we get

$$\binom{36}{2} - 3\binom{18}{2} = 630 - 3 \cdot 153 = 171.$$

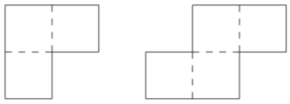
□

5. Another castle has infinitely many rooms labeled $1, 2, 3, \ldots,$ divided into several halls. Suppose room n is on the same hall as rooms $2n + 1$ and $8n + 1$ for every n. Determine the maximum possible number of different halls in the castle.

   Solution. We claim that there can be infinitely many different halls. Imagine writing the hall numbers in binary. Then going from n to $2n + 1$ means adding a "1" on the end of the binary room number, and going from n to $8n + 1$ means adding "001" on the end of the number.

Now suppose we have a room number that is 2 times an odd number, so its binary representation ends in "10." Then going from this number to a smaller number would require at some point removing the last "0," which means it would have to be part of a "001" at the end of the number, since the only things we can remove are a "001' or a "1." However, the operations of adding or removing "1" or "001" at the end of the number can only change the number of 0's occurring after the 0 we are considering by an even number, and thus never by 1. Thus, there is no way to get from this room number to a smaller room number, so it must be the first on its hall. Thus, rooms $1, 2, 6, 10, \ldots$ can each start a different hall, so there can be infinitely many halls. □

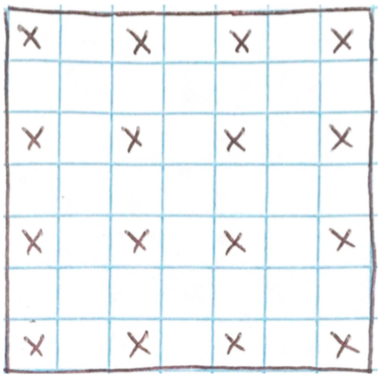
6. A 99×99 rectangular region is to be tiled with tiles of the following shape:



We allow rotations and reflections, but the tiles must remain parallel to the edges of the rectangle, and they must fit in the rectangle without overlap, covering each cell. Prove that this is possible and determine the minimum number of tiles which must be used.

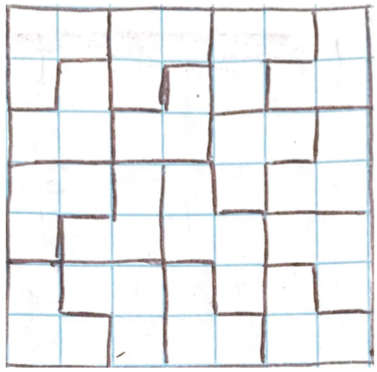
Solution. (Problem source: Putnam) In general, given a $(2m-1) \times (2n-1)$ rectangular region, the answer is mn whenever $m, n \geq 4$. We will prove the problem in this generality.

To show that this is necessary, mark all the squares that are in an odd row and odd column, as shown below:

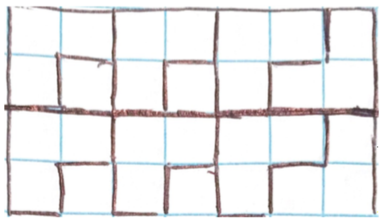


Then each tile covers at most one marked square, and the total number of marked squares is mn. Thus, at least mn tiles are needed.

To show that mn tiles is sufficient, we use induction. For the base case, with $m = n = 4$, we can see that 16 tiles are sufficient for a 7×7 grid, as shown below:



For the inductive step, suppose we can tile a $(2m-1)\times(2n-1)$ rectangle with mn tiles, similarly to the grid shown above. Now suppose we add two more rows on top, to make a $(2m+1)\times(2n-1)$ rectangle. Then we can change every 3-square tiles in the top row except the last to a 4-square tile by adding one square on top, and then add n additional 3-square tiles along the top row, as shown below:



We added n new tiles to the mn we had, so there are $mn+n=(m+1)n$ tiles to tile an $(2m+1)\times(2n-1)$ grid. We can induct on n in the same manner by adding m tiles to tile an additional two columns added on the right. Thus, we can induct on both m and n to see that mn tiles suffices for a $(2m-1)\times(2n-1)$ grid for any m and n.

□

7. Suppose there are 100 prisoners, each of whom is given a number between 1 and 100. There is also a room with 100 boxes, labeled 1 to 100, and 100 pieces of paper with the numbers 1 to 100 on them. Each piece of paper is randomly placed inside one of the 100 boxes.

   One at a time, each prisoner is allowed to enter the room and open at most 50 boxes to see the numbers inside. If every prisoner opens the box with his own number inside it, they will all be released. They are not allowed to communicate at all during this process, but they can come up with a strategy beforehand. Show that there is a strategy that gives them at least a 30% chance of winning.

   Solution. They can use the following strategy: each prisoner opens the box labeled with his own number. He then looks at the number on the paper inside and opens the box labeled with that number next. He continues doing so until he finds his own number or he has opened 50 boxes. For instance, prisoner 2 might open box 2 and find the number 5, then open box 5 and find the number 17, then open box 17, and so on.

Thus, the prisoners will win iff the permutation of paper numbers in box numbers has no cycle of length more than 50. If the permutation does have such a cycle, it has at most one, and its length could be $51, 52, \ldots, 99$, or 100. If the cycle has length n, then there are $100!/n$ possible permutations, since there are

$$100 \cdot 99 \cdot \cdots \cdot (100 - n + 1)$$

ways to choose the numbers in the cycle (in order) and $(100 - n)!$ ways to permute the remaining numbers, but then we divide by n since it doesn't matter to the permutation which of the n numbers is first when we write out the cycle. Thus, the probability there is a cycle of length $n > 50$ is $1/n$ for each n. There is at most one such cycle, so the total probability is

$$\begin{aligned} &\frac{1}{51} + \frac{1}{52} + \cdots + \frac{1}{100} \\ &= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{100}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{50}\right) \\ &\approx \ln 100 - \ln 50 = \ln 2 \approx 30\%. \end{aligned}$$

□