Berkeley Math Circle: Monthly Contest 5 Solutions

1. A *palindrome* is a positive integer that reads the same forward and backward, like 2552 or 1991. Find a positive integer greater than 1 that divides all four-digit palindromes.

   Solution. We claim that 11 divides all four digit palindromes. Note that any four digit palindrome $abba$ is the sum

   $$abba = a00a + bb0 = a \times 1001 + b \times 110.$$

   Now, 110 is a multiple of 11, since it is 11×10, and 1001 is a multiple of 11, since it is 11×91. Thus, $a00a$ and $bb0$ are multiples of 11, so their sum is also a multiple of 11. $\square$

2. A sheet of graph paper has perpendicular grid lines spaced 1 unit apart. On the paper, you draw a polygon all of whose edges lie along the grid lines. Determine all possible perimeters for this polygon.

   Solution. The possible perimeters are all even numbers greater than 2. To show that these are all possible, note that a $1 \times (n-1)$ rectangle has perimeter $1 + (n-1) + 1 + (n-1) = 2n$, so there is a shape with perimeter $2n$ for every $n \geq 2$. To show that no shape can have odd perimeter, imagine walking along the boundary of the shape starting from some point and coming back to the same point. For every step to the right, you must also take a step to the left at some point, and for every step up, you must also take a step down at some point to end up back where you started. Thus, the total number of steps must be even. $\square$

3. Suppose x, y, and z are real numbers that satisfy $x + y + z > 0$, $xy + yz + zx > 0$ and $xyz > 0$. Prove that x, y, and z must all be positive.

   Solution. (Problem source: Aops) Note that x, y, and z are the roots of the polynomial

   $$(t - x)(t - y)(y - z) = t^3 - (x + y + z)t^2 + (xy + xz + yz)t - xyz.$$

   We claim that this polynomial has no negative roots. To see this, if we plug in a negative value of t, all four terms we are adding are negative, so the sum is negative. If we plug in $t = 0$, the first three terms are negative, and $-xyz$ is negative since xyz is positive, so the sum is again negative. Thus, whenever $t \leq 0$, the sum is negative, so any real root must satisfy $t > 0$. So, since x, y, and z are all roots, they must all be positive. $\square$

4. You are blindfolded and have a spinning table with four switches on it in front of you. The switches are always either up or down, and you don't know what configuration they start in. On each move, you can spin the table some unknown amount, then

reach out and choose two switches (either next to each other or diagonally opposite each other), feel whether they are up or down, and then choose to flip one, both, or neither of them. You win if you turn the switches either all up or all down. Determine whether it is always possible to win the game.

Solution. It is indeed possible. One strategy is as follows:

a) Choose two switches that are next to each other. Turn them so they are both up.

b) Choose two switches that are diagonally opposite each other (so exactly one is a switch you picked last time). Turn them so they are both up. Now at least three switches are up, so if the fourth is also up, you win. Thus, assume the fourth switch is down.

c) Choose two switches that are diagonally opposite each other. If one is down, turn it up, and you win since all four are up. Otherwise, both are up, so turn one of them down. There are now two adjacent down switches and two adjacent up switches.

d) Choose two adjacent switches. If both are up, turn them both down, and you win. If both are down, turn them both up, and you win. Otherwise, one is up and the other down, so turn the up one down and the down one up. Now there are two diagonally opposite up switches and two diagonally opposite down switches.

e) Choose two diagonally opposite switches. If they are up, turn them down, and if they are down, turn them up. Either way, you win!

□

5. Find a formula for the sum of the squares of the numbers in the nth row of Pascal's triangle (i.e. the numbers $\binom{n}{0}$, $\binom{n}{1}$, $\ldots$, $\binom{n}{n}$).

Solution. We claim that the sum of the squares of the numbers in row n is

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n},$$

the middle number in row $2n$ of Pascal's triangle. To prove this, we can think of the right side as the number of ways to form a committee of n people chosen from some $2n$ people. For the left side, suppose the $2n$ people are n men and n women. If there are k women on the committee, then there must be $n-k$ men, so there are $\binom{n}{k}$ ways to choose the women and $\binom{n}{n-k}$ ways to choose the men, giving $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$ possible committees with k women. Adding over the possible values of k from 0 to n gives the sum on the left. Since each side counts the same thing in a different way, the two sides must be equal. □

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers $x \neq 0$,

$$f(x) + 2f\left(\frac{x-1}{x}\right) = 3x.$$

(Note that the function must be defined for all $x \in \mathbb{R}$, including $x = 0$.)

Solution. (Problem Source: Aops) Plugging $\frac{x-1}{x}$ in for x in the functional equation gives

$$f\left(\frac{x-1}{x}\right)+2f\left(\frac{\frac{x-1}{x}-1}{\frac{x-1}{x}}\right)=f\left(\frac{x-1}{x}\right)+2f\left(\frac{1}{1-x}\right)=\frac{3(x-1)}{x}.$$

Now plugging $\frac{1}{1-x}$ in for x gives

$$f\left(\frac{1}{1-x}\right)+2f\left(\frac{\frac{1}{1-x}-1}{\frac{1}{1-x}}\right)=f\left(\frac{1}{1-x}\right)+2f(x)=\frac{3}{1-x}.$$

We now have three equations in three "variables":

$$f(x)+2f\left(\frac{x-1}{x}\right)=3x, \tag{1}$$

$$f\left(\frac{x-1}{x}\right)+2f\left(\frac{1}{1-x}\right)=\frac{3(1-x)}{x}, \tag{2}$$

$$f\left(\frac{1}{1-x}\right)+2f(x)=\frac{3}{1-x}. \tag{3}$$

To isolate $f(x)$, we take equation (1) minus 2 times equation (2) plus 4 times equation (3), which makes the $f(\frac{x-1}{x})$ and $f(\frac{1}{1-x})$ terms on the left cancel and gives

$$9f(x)=3x-2\cdot\frac{3(x-1)}{x}+4\cdot\frac{3}{1-x}.$$

Dividing by 9 on both sides and combining terms on the right with a common denominator gives

$$\begin{aligned} f(x) &= \frac{x^2(1-x)+2(1-x)^2+4x}{3x(1-x)} \\ &= \frac{-x^3+3x^2+2}{3x(1-x)}. \end{aligned}$$

However, we note that this function is not defined for $x=0$ or $x=1$, while the domain of f should be all real numbers. The only constraint on $f(0)$ and $f(1)$ comes from plugging $x=1$ into the functional equation, giving

$$f(1)+2f(0)=3.$$

Thus, $f(0)$ can be any real number c, and $f(1)$ must be $3-2c$. Putting this together gives the solutions

$$f(x)=\begin{cases} \frac{-x^3+3x^2+2}{3x(1-x)} & x\neq 0,1 \\ c & x=0, \\ 3-2c & x=1 \end{cases}$$

for any real number c. □

7. An infinite castle has rooms labeled $1,2,3,\ldots$. If room n is on the same hall as rooms $2n+1$ and $3n+1$ for every n, what is the maximum possible number of different halls on the castle?

Solution. We claim that all rooms must be on the same hall. Suppose for contradiction there is a room other than room 1 that is the first room on its hall. We will show by induction that its room number must be congruent to $2^{2k-1} \pmod{3 \cdot 2^{2k-1}}$ for every positive integer k, and therefore must be infinitely big.

First, note that room $2n$ is on the same hall as room $3n$ for every n, since both are connected to room

$$3 \cdot 2n + 1 = 2 \cdot 3m + 1 = 6n + 1.$$

Thus, no room that is a multiple of 3 is first on its hall. Also, no room that is odd or that is 1 (mod 3) besides room 1 is first on its hall, since if n is odd room n is on the same hall as room $(n-1)/2$, and if n is 1 (mod 3) room n is on the same hall as room $(n-1)/3$.

Now, for the base case, we claim that a room other than room 1 that is first on its hall must be 2 (mod 6). To see this, note that the room number cannot be 1, 3, or 5 (mod 6), since then it would be odd; it cannot be 0 (mod 6), since then it would be a multiple of 3; and it cannot be 4 (mod 6), since then it would be 1 (mod 3). Thus, it must be 2 (mod 6).

For the inductive step, suppose our room number can be written as $3 \cdot 2^{2k-1} \cdot m + 2^{2k-1}$ for nonnegative integers m and k. We claim first that m is odd. Thus, suppose m is even.

By repeatedly applying the rule that room $2n$ is on the same hall as room $3n$, our room is on the same hall as room

$$\left(\frac{3}{2}\right)^{2k-1} (3 \cdot 2^{2k-1} \cdot m + 2^{2k-1}) = 3^{2k} \cdot m + 3^{2k-1}.$$

Since we are assuming m is even, this room number is odd, so it is on the same hall as room

$$\frac{3^{2k} \cdot m + 3^{2k-1} - 1}{2}.$$

This room number is 1 (mod 3), so it is on the same hall as room

$$\frac{\frac{3^{2k} \cdot m + 3^{2k-1} - 1}{2} - 1}{3} = \frac{3^{2k-1} \cdot m + 3^{2k-2} - 1}{2}.$$

Repeatedly applying the rule that room $n$is on the same hall as room $(n-1)/3$, we find that this room is on the same hall as room

$$\frac{3^{i} \cdot m + 2^{i-1} - 1}{2}$$

for every positive integer i. In particular, for $i = 1$, it is on the same hall as room $3m/2$. This is a smaller room number than the one we started with, contradicting the assumption that our room was the first on its hall. We conclude that m must be odd.

Thus, we can write $m = 2j + 1$ for some j, so our room number is now

$$3 \cdot 2^{2k-1}(2j + 1) + 2^{2k-1} = 3 \cdot 2^{2k} \cdot j + 2^{2k+1}.$$

We now claim that j is even, which will complete our induction.

Thus, suppose j is odd. As before, we repeatedly apply the rule that rooms $2n$ and $3n$ are on the same hall to find that this room is on the same hall as room

$$\left(\frac{3}{2}\right)^{2k}(3 \cdot 2^{2k} \cdot j + 2^{2k+1}) = 3^{2k+1} \cdot j + 2 \cdot 3^{3k}.$$

Since we are assuming j is odd, this number is odd, so it is on the same hall as room

$$\frac{3^{2k+1} \cdot j + 2 \cdot 3^{3k} - 1}{2}.$$

This room number is 1 (mod 3), and similarly to before, we find by repeatedly applying the rule that rooms n and $(n-1)/3$ are connected that our room is on the same hall as room

$$\frac{3^i \cdot j + 2 \cdot 3^{i-1} - 1}{2}$$

for each positive integer i. In particular, it is on the same hall as room $(3j+1)/2$, which is smaller than our starting room number. This contradicts the assumption that j was odd, so j must be even. This completes our proof.

□