Berkeley Math Circle: Monthly Contest 4 Solutions

1. Prove that $\sqrt{n+1}+\sqrt{n}$ is irrational for every positive integer n.

   Solution. Assume for contradiction that it was rational, and let q denote its value. Squaring, we find that

$$q^2 = (n+1) + 2\sqrt{n}\cdot\sqrt{n+1} + n$$

   so

$$\frac{q^2-(2n+1)}{2} = \sqrt{n(n+1)}.$$

   The left-hand side is also rational, so we conclude the quantity $n(n+1)$ is the square of a rational number. Actually, since it is an integer, it follows that $n(n+1)$ must be a perfect square (the square of an integer). However, $n^2 < n(n+1) < (n+1)^2$, so $n(n+1)$ lies strictly between two consecutive perfect squares, which is a contradiction. □

2. Suppose a sequence s_1, s_2, $\ldots$, of positive integers satisfies $s_{n+2} = s_{n+1} + s_n$ for all positive integers n (but not necessarily $s_1 = s_2 = 1$). Prove that there exists an integer r such that $s_n - r$ is not divisible by 8 for any integer n.

   Solution. We start by observing that the "classic" Fibonacci sequence goes 112350552710 modulo 8 (repeating forever with period 12), and hence takes on only 6 distinct values mod 8 (the values 4 and 6 are omitted). So the statement is true there.

   Now let $s_1 = a$ and $s_2 = b$. If this sequence never hits a multiple of 8, we are done (with $r = 0$). Otherwise, the sequence has a passage that goes $0, c, c$, which implies that mod 8 we have a sequence that is a shift of the sequence multiplied by c (since the Fibonacci sequence contains consecutive terms $0, 1, 1$). But this gives at most 6 distinct values mod 8, as needed. □

3. For positive real numbers a, b, c satisfying $ab + bc + ca = 1$, prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge a^2 + b^2 + c^2 + 2.$$

   Solution. By Cauchy-Schwarz, we have

$$(ab+bc+ca)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge (a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$$

   which implies the result immediately. □

4. Suppose we have a convex polygon in which all interior angles are integers when measured in degrees, and the interior angles at every two consecutive vertices differ by exactly 1°. If the greatest and least interior angles in the polygon are M° and m°, what is the maximum possible value of $M - m$?

Solution. The answer is 18.

To justify this answer, we will find it helpful to discuss the exterior angles rather than the interior angles. Consecutive exterior angles must still be integers and must still differ by 1°, and the value we seek is equal to the difference between the greatest and least *exterior* angles (since $(180 - m) - (180 - M) = M - m$). But by working with the exterior angles, we gain one useful fact: they must add up to 360°.

We can achieve $M - m = 18$ by letting the exterior angles in a 36-gon be

$$1^\circ, 2^\circ, 3^\circ, \ldots, 18^\circ, 19^\circ, 18^\circ, \ldots, 3^\circ, 2^\circ$$

in that order. The sum is 360° (since we have 36 angles whose average is 10°), and such a polygon clearly exists (we can construct a convex polygon with prescribed exterior angles $a_1, \ldots, a_n$ by letting the vertices be $(1, a_1 + \cdots + a_k)$ in polar coordinates for all $1 \le k \le n$).

Now we show that we cannot achieve $M - m > 18$. We argue by contradiction. Suppose $M - m > 18$. Then $M - m \ge 19$ and $M \le 179$, so $m \le 160$, and the greatest exterior angle is at least $180^\circ - 160^\circ = 20^\circ$. An exterior angle k vertices away from the greatest exterior angle must be at least $(180 - m - k)^\circ$. Two facts follow: first, there must be a vertex at least 19 vertices away from the vertex with the greatest exterior angle, and so there are at least 38 vertices; second, the exterior angles in the polygon add up to at least

$$1^\circ + 2^\circ + 3^\circ + \cdots + 19^\circ + 20^\circ + 19^\circ + \cdots + 3^\circ + 2^\circ,$$

which is 399°. This is a contradiction.

Therefore, the maximum possible value of $M - m$ is 18. □

5. Given a quadrilateral $ABCD$ extend AD and BC to meet at E and AB and DC to meet at F. Draw the circumcircles of triangle ABE, ADF, DCE, and BCF. Prove that all four of these circles pass through a single point.

Solution. Let circumcircles of BCF and DCE intersect at point M. We find $\angle DMF$. As $\angle DMC = \angle DEC$, and $\angle MCF = 180 - \angle CBF$, we have $\angle DMF = 180 - (\angle CBF - \angle DEC) = 180 - \angle EAB$, so $ADMF$ are concyclic. Similarly, $ABME$ are concyclic. □

6. Determine, with proof, whether or not there exist distinct positive integers a_1, a_2, $\ldots$, a_n such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 2019.$$

Solution. Yes, the decomposition exists.

Recall that the harmonic series diverges. We first take the largest partial sum of the harmonic series that is smaller than 2019, subtract it from 2019 to get a "remainder" r. We then use the greedy algorithm to pick the rest of the unit fractions: pick the largest integer n with $1/n \le r$, and replace r with $r - 1/n$. It is not hard to see the integers chosen at each step increase.

The main observation is that as we whittle away at the remainder r, the numerator of the remainder decreases at every step. Therefore this process must eventually terminate, and it can only terminate when $r = 0$, as needed. □

7. A simple graph G on 2020 vertices has its edges colored red and green. It turns out that any monochromatic cycle has even length. Given this information, what is the maximum number of edges G could have?

   Solution. Note that G has no K_5; indeed, it's well-known that the only triangle-free coloring of the edges of K_5 consists of two monochromatic 5-cycles. Therefore, the number of edges of G is at most $\binom{4}{2} \cdot 505^2 = 1530150$ by Turán's theorem.

   To show this occurs, we split the graph into four equally sized components A, B, C, D. We color red all edges between A and B, or C and D. We color green all edges between A and C, A and D, B and C, and B and D. This indeed has the claimed number of edges, and the subgraphs formed by each color are bipartite, so this solves the problem. □