Berkeley Math Circle: Monthly Contest 2 Solutions

1. The five-digit number $9A65B$ is divisible by 36, where A and B are digits. Find all possible values of A and B.

   SOLUTION. A number is divisible by 36 iff it is divisible by 9 and 4. For it to be divisible by 4, the last two digits must make a 2-digit number divisible by 4, and the only possibilities for this are $B = 2$ or $B = 6$. For it to be divisible by 9, the sum of the digits must be a multiple of 9. Right now the sum is $9+6+5 = 20$, and the next multiple of 9 is 27, so we need $A + B = 7$. Thus, the two solutions are $A = 5, B = 2$ or $A = 1, B = 6$.

2. If x is a positive real number, find the smallest possible value of $2x + \frac{18}{x}$.

   SOLUTION. The answer is 12. This is achieved when $x = 3$; to see it is optimal, note that $2x + \frac{18}{x} \geq 6 \iff x^2 - 6x + 9 \geq 0$, which is obviously true since the left-hand side is $(x - 3)^2 \geq 0$.

   Alternatively, those who know the so-called AM-GM inequality may apply it directly.

3. There are infinitely many bowls arranged on the number line, one at each integer. Initially each bowl has one fruit in it. In a move, one may take any fruit and move it to an adjacent bowl (bowls may hold more than one fruit, or no fruits at all).

   Is it possible that after 999 moves, every bowl still has exactly one fruit remaining?

   SOLUTION. The answer is no.

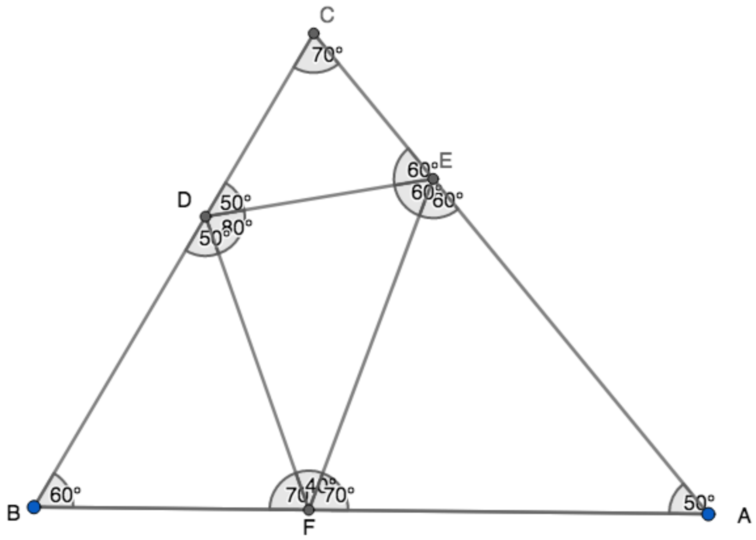
   After 999 moves, only finitely bowls were touched (i.e. received or lost any fruit), so we can consider a large interval $[L, M]$ such that all touched bowls are in this interval.

   Now, consider the sum S of the indices of all fruits inside this interval. By hypothesis, S is the same before and after the moves. However, every move changes the parity of S, so after 999 moves the value of S should have opposite parity, contradiction.

4. In triangle ABC, $\angle A = 50°, \angle B = 60°$, and $\angle C = 70°$. A ray of light bounces from point D on BC to E on CA to F on AB and then back to D. Find the angles of $\triangle DEF$.

   (Light always takes the shortest path between points, meaning it bounces off an edge at equal angles.)

   SOLUTION. Let $\angle BDF = \angle CDE = x$, $\angle CED = \angle AEF = y$, and $\angle AFE = \angle BFD = z$.



Since $\triangle AEF$, $\triangle BDF$, and $\triangle CDE$ all have angle sum $180°$, we get

$$x + y + 70° = 180°,$$
$$x + z + 60° = 180°,$$
$$y + z + 50° = 180°.$$

Adding the three equations, subtraction $180°$ from both sides, and dividing by 2 gives

$$x + y + z = 180°.$$

From this, we get

$$x = 50°, y = 60°, z = 70°.$$

(Thus, the triangles $\triangle AEF$, $\triangle BDF$, and $\triangle CDE$ are in fact all congruent to $\triangle ABC$!) Now we get

$$\angle EDF = 180° - 2 \cdot 50° = 80°,$$
$$\angle DEF = 180° - 2 \cdot 60° = 60°,$$
$$\angle DFE = 180° - 2 \cdot 70° = 40°.$$

5. Aerith and Bob take turns picking a nonnegative integer, each time subtracting a (positive) divisor from the other's last number. The first person to pick 0 loses. For example, if Aerith reached 2020 on some turn, Bob could pick $2020 - 20 = 2000$, as 20 is a divisor of 2020.

   Continuing this example (with Aerith now picking a divisor of 2000), if both of them play optimally, who wins?

   SOLUTION. If $1998 = 2000 - 2$ were a losing position, Aerith would pick it. Otherwise, she can choose $1999 = 2000 - 1$. and since this is prime, Bob must pick either $1999 - 1999 - 0$ or $1999 - 1 = 1998$. In any case, Aerith wins.

6. Find all perfect squares that can be written as the sum of two powers of 2.

SOLUTION. We claim that the squares which work are 4^{k+1} or $9 \cdot 4^k$ for $k \in \mathbb{Z}_{\geq 0}$. These work because $4^{k+1} = 2^{2k+1} + 2^{2k+1}$ and $9 \cdot 4^k = 2^{2k+3} + 2^{2k}$.

Write $n^2 = 2^a + 2^b$ for $n, a, b \in \mathbb{Z}_{\geq 0}$ and $a \geq b$. If $a = b$, $n^2 = 2^{b+1}$, and since the perfect squares that are powers of two are the powers of 4, we have $n = 4^{k+1}$ for some $k \in \mathbb{Z}_{\geq 0}$.

Otherwise, $a - b > 0$ so 2^{a-b} is even and $2^{a-b} + 1$ is odd. Therefore, 2^b and $2^{a-b} + 1$ are relatively prime, so since $n^2 = 2^a + 2^b = 2^b(2^{a-b} + 1)$, 2^b and $2^{a-b} + 1$ are perfect squares, so 2^b is a power of 4. Let $m^2 = 2^{a-b} + 1$, so $(m-1)(m+1) = 2^{a-b}$. Then, $m - 1$ and $m + 1$ are powers of 2 that differ by 2. By inspection, we see that $m = 3$ so $2^{a-b} + 1 = 9$ and n^2 is 9 times a power of 4, as desired.

7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$(x - y)f(x + y) = xf(x) - yf(y).$$

SOLUTION. We claim that the answer is $f(t) = mt + b$ for $m, b \in \mathbb{R}$. This works because

$$\begin{aligned} m(x+y)(x-y) + b(x-y) &= m\left(x^2 - y^2\right) + bx - by \\ (x-y)(m(x+y)+b) &= x(mx+b) - y(my+b). \end{aligned}$$

We now show that only $mt + b$ works. Let $P(x, y)$ denote the problem equation. Taking $P(1, t-1) + P(t-1, -t) + P(-t, t)$ we get

$$\begin{array}{rr} P(1,t-1): & (2-t)f(t) = f(1) - \cancel{(t-1)f(t-1)} \\ P(t-1,-t): & (2t-1)f(-1) = \cancel{(t-1)f(t-1)} + \cancel{tf(-t)} \\ + \quad P(-t,t): & -2tf(0) = \cancel{-tf(-t)} - tf(t) \\ \hline \multicolumn{2}{r}{(2-t)f(t) + (2t-1)f(-1) - 2tf(0) = f(1) - tf(t)} \\ & 2f(t) = f(1) + 2tf(0) + (1-2t)f(-1) \\ & f(t) = (f(0) - f(-1))t + \frac{f(1)+f(-1)}{2}, \end{array}$$

so taking $m = f(0) - f(-1), b = \frac{f(1)+f(-1)}{2}$ gives $f(t) = mt + b$, as desired.