Berkeley Math Circle: Monthly Contest 1 Solutions

1. A scalene triangle has side lengths which are all prime numbers. What is the smallest possible perimeter it could have?

   SOLUTION. The answer is $3+5+7=15$. This is achievable since the three sides $\{3,5,7\}$ form the sides of a triangle.

   To show it is best possible, note that $\{3,5,7\}$ are the three smallest odd primes; so any smaller triangle would need to have a side of length 2. However, the two longer lengths would then be different odd primes; this would cause them to differ by at least 2, which is impossible.

2. In the addition equation

$$\begin{array}{cccc} & B & M & C \\ + & & M & C \\ \hline X & Y & 7 & 0 \end{array}$$

   each letter stands for a distinct digit and leading digits are not 0. Determine which digit each letter stands for.

   SOLUTION. Since the 4-digit number $XY70$ is less than 100 more than the 3-digit number BMC, we must have $B=9$, $X=1$, and $Y=0$. Since $C+C$ gives a 0 in the 1's place and C cannot be the same as Y, we must have $C=5$. We carry a 1 from the 1's place to the 10's place, so $M+M+1$ gives a 7 in the 1's place. We must also carry a 1 from the 10's place to the 100's place, so $M+M+1=17$, which gives $M=8$.

   Thus, we have $BMC=985$, $MC=85$, and $XY70=1070$.

3. Aerith has written down two letters on a piece of paper. Bob will pick a positive integer and she'll try to factor Bob's positive integer into two others, such that when spelled in English, one contains her first letter and the other her second.

   For example, if Aerith had chosen letters "v" and "w" and Bob chose "ten", she could spell it out as "fi<u>v</u>e" times "t<u>w</u>o," but would fail if Bob chose "twenty".

   Show that there is a unique pair of letters Aerith can choose that allows her to succeed no matter what number Bob picks.

   SOLUTION. We can narrow down her possible choices quickly:

   - If Bob chose "one," Aerith would have to have to say "one times one", so her two letters must both be in the word "one".
   - If Bob chose "three," she would have to say "one times three", so one of her letters must be in "three" and thus must be "e".

- If Bob chose "four," she could not have said "two times two" as neither word contains "e", so she must have said "one times four". Thus her second letter must be "o".

Thus, Aerith has letters "o" and "e".

We now show that all positive integers can be represented this way. We note that any number containing either one of "o" or "e" works, because multiplying itself by "one" will supply the other letter. As such,

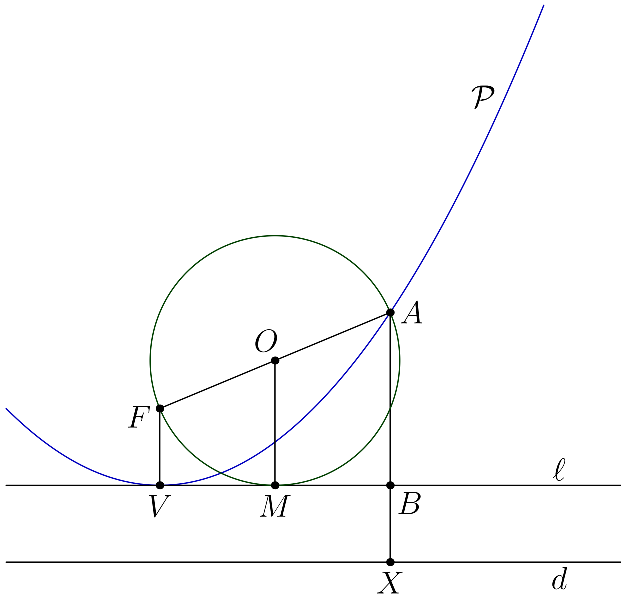
- any number 1000 or higher works because it must contain "o" ("th$\underline{\text{o}}$usand", "milli$\underline{\text{o}}$n", "billi$\underline{\text{o}}$n", ...), and
- any number from 100 to 999 works, because it must contain "hundr$\underline{\text{e}}$d",
- 10 through 19 work, because all of "ten", "eleven", "twelve", and "thirteen" through "nineteen" contain "e",
- any other number ending in digits other than 0 or 6 works, because all of "one", "two" "three", "four" "five", "seven", "eight" and "nine" also contain either "e" or "o",
- all numbers ending in 6 can be written as "two" times a number ending in "three", "eight", "thirteen" or "eighteen", all of which contain "e",
- 30, 50, and 60 can be factored as "tw$\underline{\text{o}}$" times "fift$\underline{\text{e}}$en", "twentyfiv$\underline{\text{e}}$", or "twentyfiv$\underline{\text{e}}$", respectively, and
- the remaining multiples of 10, i.e., "tw$\underline{\text{e}}$nty", "f$\underline{\text{o}}$rty", "s$\underline{\text{e}}$venty", "$\underline{\text{e}}$ighty" and "nin$\underline{\text{e}}$ty", all contain either "e" or "o".

This covers all natural numbers, as desired.

4. Let F be a point, d be a line, and let $\mathcal{P}$ be the parabola with focus F and directrix d, i.e. the set of points which are equidistant from F and d. Show that there is a line ℓ such that for any point A on $\mathcal{P}$, the circle with diameter AF is tangent to ℓ.

   SOLUTION. Let

   - ℓ be the line parallel to d equidistant from F and d,
   - V be the vertex of $\mathcal{P}$, i.e., the foot of the altitude from F to ℓ,
   - X be the foot of the altitude from A to d,
   - O be the midpoint of $\overline{FA}$,
   - B be the foot of the altitude from A to ℓ, and
   - M be the midpoint of $\overline{BV}$, i.e., the foot of the altitude from O to ℓ.



We have $AB + XB = AX$. By definition of $\mathcal{P}$, $AX = AF$, and by definition of ℓ, $XB = FV$, so $AB + FV = AF$. Since OM is the average of FV and AB, this means that $OM = AF/2$. Thus the circle with diameter AF has center O and radius OM, as desired.

5. Let a, b, c be positive real numbers with $a + b + c = 1$. Prove that

$$a^4 + b^4 + c^4 \geq abc.$$

SOLUTION. Since $a + b + c = 1$, we can multiply the right side by $a + b + c$ to get the equivalent inequality

$$a^4 + b^4 + c^4 \geq a^2bc + ab^2c + abc^2.$$

By AM-GM,

$$\frac{2a^4 + b^4 + c^4}{4} = \frac{a^4 + a^4 + b^4 + c^4}{4} \geq \sqrt[4]{a^4a^4b^4c^4} = a^2bc.$$

Similarly,

$$\frac{a^4 + 2b^4 + c^4}{4} \geq ab^2c$$
$$\frac{a^4 + b^4 + 2c^4}{4} \geq abc^2.$$

Adding these three inequalities proves the desired inequality.

6. Aerith writes 50 consecutive positive integers in a circle on a whiteboard. Each minute after, she simultaneously replaces each number x with $2020a - x + 2020b$,

where a and b were the numbers next to x. Can she choose her initial numbers such that she will never write down a negative number?

SOLUTION. Let R_0 be the initial sum of every other number that Aerith wrote down, and let S_0 be the sum of the remaining numbers. Let R_t and S_t be the corresponding sums after t minutes. Because $D = R_0 + S_0$ is a sum of 25 even and 25 odd numbers, it is itself odd, so $R_0 - S_0 \neq 0$.

We have

$$R_t = 2020S_{t-1} - R_{t-1} + 2020S_{t-1} = 4040S_{t-1} - R_{t-1}$$

and similarly

$$S_t = 4040S_{t-1} - R_{t-1}.$$

Adding and subtracting, we get

$$R_t + S_t = 4039(R_{t-1} + S_{t-1})$$

and

$$R_t - S_t = -4041(R_{t-1} - S_{t-1}),$$

so $|R_t - S_t|$ grows exponentially faster than $R_t + S_t$. Thus, for some T, $|R_T - S_T| > R_T + S_T$. This could not be true if R_T and S_T were both nonnegative, so Aerith will eventually write down a negative number.

7. Given a fixed triangle $\triangle ABC$ and a point P, find the maximum value of

$$\frac{AB^2 + BC^2 + CA^2}{PA^2 + PB^2 + PC^2}.$$

SOLUTION. We use the following lemma.

> **Lemma**. Given $a, b, c, p \in \mathbb{R}$ such that not all of a, b, c are equal,
>
> $$(a-b)^2 + (b-c)^2 + (c-a)^2 \leq 3((p-a)^2 + (p-b)^2 + (p-c)^2),$$
>
> with equality if and only if $p = (a+b+c)/3$.

Proof. For fixed a, b, c, the right hand side is a quadratic in p, namely

$$\begin{aligned} 3((p-a)^2 + (p-b)^2 + (p-c)^2) &= 3((p^2 - 2ap + a^2) + (p^2 - 2bp + b^2) + (p^2 - 2cp + c^2)) \\ &= 9p^2 - 6(a+b+c)p + 3(a^2 + b^2 + c^2). \end{aligned}$$

and is thus minimized at $-\frac{-6(a+b+c)}{2\cdot 9} = (a+b+c)/3$. By plugging $p = (a+b+c)/3$ into the original equation, we observe equality, as desired. □

Using Cartesian coordinates, let $A = (a_x, a_y)$, $B = (b_x, b_y)$, $C = (c_x, c_y)$, and $P = (p_x, p_y)$. By the Pythagorean theorem,

$$\begin{aligned} AB^2 + BC^2 + CA^2 &= \left[(a_x - b_x)^2 + (a_y - b_y)^2\right] + \left[(b_x - c_x)^2 + (b_y - c_y)^2\right] \\ &\quad + \left[(c_x - a_x)^2 + (c_y - a_y)^2\right] \\ &= \left[(a_x - b_x)^2 + (b_x - c_x)^2 + (c_x - a_x)^2\right] \\ &\quad + \left[(a_y - b_y)^2 + (b_y - c_y)^2 + (c_y - a_y)^2\right], \end{aligned}$$

so by the above,

$$
\begin{aligned}
AB^2 + BC^2 + CA^2 &\leq 3\left[(p_x - a_x)^2 + (p_x - b_x)^2 + (p_x - c_x)^2\right] \\
&\quad + 3\left[(p_y - a_y)^2 + (p_y - b_y)^2 + (p_y - c_y)^2\right] \\
&= 3\left[(p_x - a_x)^2 + (p_y - a_y)^2\right] + 3\left[(p_x - b_x)^2 + (p_y - b_y)^2\right] \\
&\quad + 3\left[(p_x - c_x)^2 + (p_y - c_y)^2\right] \\
&= 3(PA^2 + PB^2 + PC^2).
\end{aligned}
$$

This gives

$$\frac{AB^2 + BC^2 + CA^2}{PA^2 + PB^2 + PC^2} \leq 3,$$

with equality at

$$P = \left(\frac{a_x + b_x + c_x}{3}, \frac{a_y + b_y + c_y}{3}\right),$$

the centroid of $\triangle ABC$.