Berkeley Math Circle: Monthly Contest 8 Solutions

1. Find an example of a triangle ABC with integer side lengths such that if M is the midpoint of $\overline{BC}$ and D is the foot of the altitude from A to $\overline{BC}$, then AD and AM have integer lengths too.

   Solution. One example is to take a right triangle with $AB = 30$, $AC = 40$, $BC = 50$. Since the triangle is right, it follows that $AM = \frac{1}{2}BC = 25$ (as the midpoint M of $\overline{BC}$ is the circumcenter). Moreover, as the triangle has are $\frac{1}{2}(30)(40) = 600$, the altitude AD satisfies $\frac{1}{2}AD \cdot BC = 600$, so $AD = 24$. $\square$

2. Find all pairs (m, n) of integers which satisfy the equation $m^2 + m = n^2 - 2n$.

   Solution. The answers are the four pairs $(0, 0)$, $(0, 2)$, $(-1, 0)$ and $(-1, 2)$. These four plainly work, so we prove they are the only ones.

   Multiplying both sides by 4, we get $4n^2 - 8n = 4m^2 + 4m$ which rearranges to

   $$4n^2 - 8n + 4 = (4m^2 + 4m + 1) + 3 \implies (2n - 2)^2 - (2m + 1)^2 = 3.$$

   The only squares which differ by 3 are $2^2 = 4$ and $1^2 = 1$ (because if $x^2 - y^2 = 3$ for $x, y > 0$, then $3 = (x - y)(x + y)$, hence $x - y = 1$ and $x + y = 3$). Hence we must have $2n - 2 = \pm 2$ and $2m + 1 = \pm 1$. So $n \in \{0, 2\}$ and $m \in \{0, -1\}$, as we claimed. $\square$

3. Show that the numbers from 1 to 99 can be partitioned into two sets A and B with equal sum, and which satisfy $|A| = |B| + 1$ (i.e. A has one more element than B).

   Solution. One possible construction is:

   $$A = \{1, 2\} \cup \{4, 7, 8, 11, 12, 15, 16, 19, \dots, 96, 99\}$$
   $$B = \{3\} \cup \{5, 6, 9, 10, 13, 14, 17, 18, \dots, 97, 98\}.$$

   Since $4 + 7 = 5 + 6$, $8 + 11 = 9 + 10$, and so on, as well as $1 + 2 = 3$, this fulfills both conditions. $\square$

4. Let $s(n)$ denote the sum of the digits of a positive integer n. For example, $s(2019) = 2 + 0 + 1 + 9 = 12$. Find the 4-digit number n such which minimizes $\frac{n}{s(n)}$. (Leading zeros are not permitted.)

   Solution. If the number is $n = \overline{abcd} = 1000a + 100b + 10c + d$, we seek to minimize

   $$r = \frac{1000a + 100b + 10c + d}{a + b + c + d} = 1 + \frac{999a + 99b + 9c}{a + b + c + d}.$$

   From this expression, we see that setting $d = 9$ is optimal, in the sense that if $d \neq 9$ then replacing d with 9 will decrease the value of r.

Then, we write

$$r = \frac{1000a + 100b + 10c + d}{a + b + c + d} = 10 + \frac{990a + 90b - 9 \cdot 9}{a + b + c + 9}.$$

Since the numerator is still necessarily positive, by the same logic, replacing $c = 9$ will decrease r.

Next, we write

$$r = \frac{1000a + 100b + 10c + d}{a + b + c + d} = 1000 + \frac{-900b - 990 \cdot 9 - 999 \cdot 9}{a + b + 9 + 9}.$$

As the numerator is now necessarily negative, this sum is minimized when a is as small as possible, hence take $a = 1$.

Finally, we write

$$r = \frac{1000a + 100b + 10c + d}{a + b + c + d} = 100 + \frac{900 \cdot 1 - 90 \cdot 9 - 99 \cdot 9}{1 + b + 9 + 9}.$$

As the numerator is still negative, by the same logic, replacing $b = 0$ will decrease r.

In conclusion, the best possible value of n must be $n = 1099$. □

5. Suppose 4951 distinct points in the plane are given such that no four points are collinear. Show that it is possible to select 100 of the points for which no three points are collinear.

Solution. This is an example of a direct greedy algorithm: we will simply grab points until we are stuck.

Consider a maximal set S of the points as described (meaning no more additional points can be added), and suppose $|S| = n$. Then the $4951 - n$ other points must each lie on a line determined by two points in S, meaning

$$4951 - n \le \binom{n}{2} \implies n + \binom{n}{2} \ge 4951.$$

This requires $n \ge 100$. □

6. For which integers $n \ge 3$ does there exist a convex equiangular n-gon with rational side lengths which is not regular?

Solution. For any composite n.

First we prove prime $n = p$ are not okay. Letting $a_0, \ldots, a_{p-1}$ be side lengths and ζ a primitive pth root of unity, so that

$$0 = \sum_{i=0}^{p-1} a_i \zeta^i.$$

But the minimal polynomial of ζ is the cyclotomic polynomial $X^{p-1} + \cdots + 1$. So $a_0 = a_1 = \cdots = a_{p-1}$ follows.

If n is composite, let $n = pk$ for $k \ge 2$, p prime. Let $\theta = 2\pi \cdot \frac{n-2}{n}$ be the "correct" interior angle. Construct a convex polygon $\mathcal{A}$, say $A_0 A_1 \ldots A_k$, such that

- $\angle A_1 = \cdots = \angle A_{k-1} = \theta$,
- $A_0A_1 = 2$, and
- $A_iA_{i+1} = 1$ for $i = 1, 2, \ldots, k-1$.

Then on the sides of a regular p-gon (or a line segment if $p = 2$), one can erect copies of $\mathcal{A}$. That produces the desired construction. $\square$

7. Three distinct circles Ω_1, Ω_2, Ω_3 cut three common chords concurrent at X. Consider two distinct circles Γ_1, Γ_2 which are internally tangent to all Ω_i. Prove that X lies on the line joining the centers of Γ_1 and Γ_2.

   Solution. We prove that X is the exsimilicenter of the two circles. Let the chords be A_1A_2, B_1B_2, C_1C_2. Then *negative inversion* at X with power

$$XA_1 \cdot XA_2 = XB_1 \cdot XB_2 = XC_1 \cdot XC_2$$

   fixes Ω_1, hence Ω_2 and Ω_3.

   Hence this inversion swaps swaps Γ_1 with Γ_2. But if a negative inversion sends one circle to another then it is their exsimilicenter, as desired. $\square$