Berkeley Math Circle: Monthly Contest 7 Solutions

1. Several weights are given, each of which is not heavier than 1 lb. It is known that they cannot be divided into two groups such that the weight of each group is greater than 1 lb. Find the maximum possible total weight of these weights.

   Solution. Suppose you start putting weights into two pans of a scale and stop when you cannot add another weight to either pan without it exceeding 1 lb. At this point, you have put at most 2 lb of weight on the scale, and there can only be one weight left, otherwise you could add a weight to each pan and both would exceed 1 lb. This remaining weight weighs at most 1 lb, so the weights add up to at most 3 lb total. It is possible to have exactly 3 lb with three 1 lb weights, so this is the largest possible. □

2. Find the value of the infinite continued fraction

   $$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\cdots}}}}.$$

   Solution. Let the value of the fraction be x. Then we get

   $$x = 1 + \frac{1}{x+1} \implies x(x+1) = (x+1)+1 \implies x^2 = 2.$$

   Thus, since x is positive, we must have $x = \sqrt{2}$. □

3. Let C be a circle with center at the origin O of a system of rectangular coordinates, and let MON be the quarter circle of C in the first quadrant. Let PQ be an arc of C of fixed length that lies in the arc MN. Let K and L be the feet of the perpendiculars from P and Q to ON, and let V and W be the feet of the perpendiculars from P and Q to OM, respectively. Let A be the area of trapezoid $PKLQ$ and B the area of trapezoid $PVWQ$. Prove that $A+B$ does not depend on where arc PQ is chosen.

   Solution. Draw OP, OU, and OQ, and note that $A+B$ is the area of rectangle $PVWU$, plus the area of rectangle $UKLQ$, plus twice the area of the triangle PUQ. But the area of triangle POU is half the area of rectangle $PVWU$, and the area of triangle UOQ is half the area of rectangle $UKLQ$, so putting this together $A+B$ is twice the area of triangle POQ, which depends only on the fixed length of arc PQ. □

4. Prove that each nonnegative integer can be represented in the form $a^2 + b^2 - c^2$, where a, b, c are positive integers with $a < b < c$.

   Solution. We note that

   $$0 = 3^2 + 4^2 - 5^2, \quad 2 = 5^2 + 11^2 - 12^2,$$

and for $n > 1$, we can write $2n$ as

$$2n = (3n)^2 + (4n-1)^2 - (5n-1)^2,$$

and $3n < 4n-1 < 5n-1$. For odd numbers greater than 7, we can use

$$2n+3 = (3n+2)^2 + (4n)^2 - (5n+1)^2,$$

and for the first four odd numbers we have

$$1 = 4^2 + 7^2 - 8^2, \quad 3 = 4^2 + 6^2 - 7^2,$$

$$5 = 4^2 + 5^2 - 6^2, \quad 7 = 6^2 + 14^2 - 15^2.$$

This covers all cases, so the statement holds for all nonnegative integers. □

5. Let p and q be positive real numbers with $p + q < 1$. Teams A and B play a series of games. For each game, A wins with probability p, B wins with probability q, and they tie with probability $1-p-q$. The series ends when one team has won two more games than the other, that team being declared the winner of the series. What is the probability that A wins the series?

Solution. We can break the series into rounds, where each round is a series of ties followed by one team winning a game. Thus, a round that A wins has the form $A, tA, ttA, tttA, \dots$, where t represents a tied game and A a game A wins. Thus, the probability A wins a round is the geometric series

$$p' = p + p(1-p-q) + p(1-p-q)^2 + \cdots = \frac{p}{p+q}.$$

Similarly, the probability that B wins a round is $q/(p+q)$. Now, the series must last an even number of rounds, since one team must be ahead by 2. If it lasts $2n$ rounds, then for the first $2n-2$ rounds, the teams must alternate winning. The probability that each team wins one rounds from an adjacent pair is $p'q' + q'p' = 2p'q'$. This must happen $n-1$ times, and then A must win the last two rounds, with probability $(p')^2$. Thus, the probability A wins the series is another geometric series

$$\sum_{n=1}^{\infty} (2p'q')^{n-1}(p')^2 = \frac{(p')^2}{1-2p'q'} = \frac{p^2}{(p+q)^2 - 2pq} = \frac{p^2}{p^2+q^2}.$$

□

6. If triangle ABC has perimeter 2, prove that not all its altitudes can exceed $1/\sqrt{3}$ in length.

Solution. Let a, b, c be the side lengths of the triangle, where a is the longest side, so $a \geq 2/3$, and let h_a be the length of the altitude to that side. The semiperimeter is 1, so by Heron's formula we get

$$\frac{1}{2}ah_a = \sqrt{(1-a)(1-b)(1-c)} \implies h_a \leq 3\sqrt{(1-a)(1-b)(1-c)}.$$

But by AM-GM,

$$\frac{(1-a)+(1-b)+(1-c)}{3} = \frac{1}{3} \geq \sqrt[3]{(1-a)(1-b)(1-c)} \implies \sqrt{(1-a)(1-b)(1-c)} \leq \frac{1}{3\sqrt{3}}.$$

Thus, we get

$$h_a \leq 3 \cdot \frac{1}{3\sqrt{3}} \leq \frac{1}{\sqrt{3}}.$$

□

7. Let $\sigma(n)$ denote the sum of the positive divisors of n. We say n is perfect if $\sigma(n) = 2n$. If n is a positive integer such that

$$\frac{\sigma(n)}{n} = \frac{5}{3},$$

show that $5n$ is an odd perfect number.

Solution. Note that $\sigma(n)/n$ is the sum of the reciprocals of the divisors of n. Since $3\sigma(n) = 5n$, n must be divisible by 3. If n were even, then n would be divisible by 6, so we would get

$$\frac{\sigma(n)}{n} \geq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2 > \frac{5}{3},$$

a contradiction. Thus, n is odd, so $\sigma(n)$ is odd as well. This means n is a perfect square, since $\sigma(n)$ is the sum of the (odd) divisors of n, and n has an odd number of divisors if and only if it is a square. Thus, since n is divisible by 3, it is also divisible by 9. Now we claim that n is not divisible by 5. If so, it would also be divisible by 45, so we would get

$$\frac{\sigma(n)}{n} \geq 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{15} + \frac{1}{45} = \frac{78}{45} = \frac{26}{15} > \frac{5}{3}.$$

Thus, n is not divisible by 5, so $\sigma(5n) = \sigma(5)\sigma(n) = 6\sigma(n)$. Thus we get

$$\frac{\sigma(5n)}{5n} = \frac{6}{5} \cdot \frac{\sigma(n)}{n} = \frac{5}{6} \cdot \frac{5}{3} = 2.$$

Thus, $5n$ is a perfect number, and since n is odd, $5n$ is odd as well. □