Berkeley Math Circle: Monthly Contest 6 Solutions

1. Three friends wish to divide five different tasks among themselves, such that every friend must handle at least one task. In how many different ways can this be done?

   Solution. If there were no restriction that all friends are assigned a task, the number of ways to assign would simply be $3^5 = 243$. So we will use complementary counting.

   Call the friends A, B, and C. The number of ways to assign the tasks such that A and B have at least one task, but C has no tasks, is $2^5 - 2 = 30$. This is obtained by noting that one has two choices of who to give each task, subtracting off the two possibilities where A receives all tasks or B receives two tasks.

   Similarly, the number of ways to assign the tasks such that B and C have at least one task, but A has no tasks, is $2^5 - 2 = 30$.

   Similarly, the number of ways to assign the tasks such that C and A have at least one task, but B has no tasks, is $2^5 - 2 = 30$.

   Finally, there are three ways to assign all tasks to one person.

   In summary, the answer should be $3^5 - 3 \cdot 30 - 3 \cdot 1 = 150$. □

2. Let T be a triangle with area 1. We let T_1 be the medial triangle of T, i.e. the triangle whose vertices are the midpoints of sides of T. We then let T_2 be the medial triangle of T_1, T_3 the medial triangle of T_2, and so on. What is the sum of the areas of T_1, T_2, T_3, T_4, ...?

   Solution. In general, the medial triangle has side length half the original triangle, hence $\frac{1}{4}$ the area. Thus, T_1 has area $(1/4)$; then T_2 has area $(1/4)^2$, and so on. Thus the answer is

   $$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{1/4}{1 - 1/4} = \frac{1}{3}.$$

   □

3. Prove that if x, y, z are positive real numbers, then

   $$x^2 + 2y^2 + 3z^2 > xy + 3yz + zx.$$

   Solution. We note the identity

   $$2(x^2+2y^2+3z^2)-2(xy+3yz+zx) = (x-y)^2+(x-z)^2+3(y-z)^2+2z^2 \geq 2z^2 > 0.$$

   since all squares of real numbers are nonnegative. □

4. Can the sum of three fourth powers end with the four digits 2019? (A fourth power is an integer of the form n^4, where n is an integer.)

   Solution. No, in fact it cannot even end in the digit 9. The possible last digits of a fourth power are 0, 1, 5, 6. No combination of three of these add up to 9. □

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any real numbers x and y,

$$f(x+y) = \max(f(x), y) + \min(f(y), x).$$

Solution. We claim the only answer is the identity function $f(x) \equiv x$, which clearly works.

Let $(x, y) = (a, 0)$ and $(x, y) = (0, a)$ so that

$$\begin{aligned} f(a) &= \max(f(a), 0) + \min(f(0), a) \\ f(a) &= \max(f(0), a) + \min(f(a), 0). \end{aligned}$$

Sum:

$$2f(a) = (f(a) + 0) + (f(0) + a)$$

so $f(a) = a + f(0)$.

It is easy to check from here f is identity. □

6. We are given a family $\mathcal{F}$ of functions from the set $\{1, \ldots, n\}$ to itself. A sequence $(f_1, \ldots, f_k)$ of functions in $\mathcal{F}$ is said to be *good* if $f_k \circ f_{k-1} \circ \cdots \circ f_1$ is a constant function. Prove that if there exists a good sequence, there exists one with $k \leq n^3$.

Solution. Suppose there exist a good sequence.

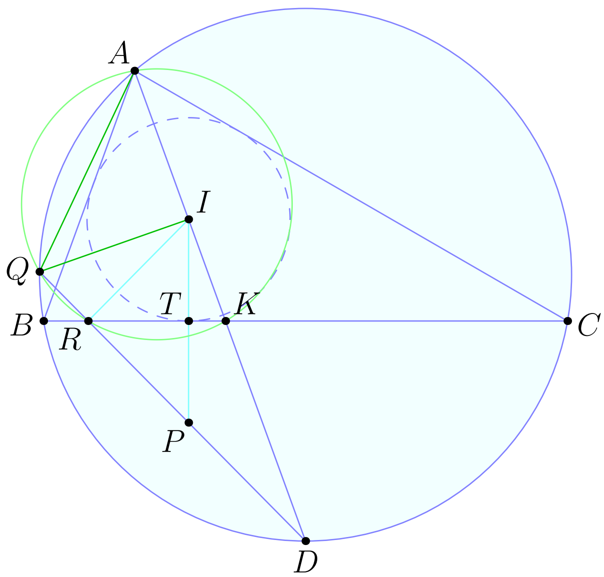
Then for any two $a, b \in \{1, \ldots, n\}$ then there is some sequence of functions $g_1, \ldots, g_m$ such that $g_m \circ g_{m-1} \circ \cdots \circ g_1$ maps a and b to the same point. Looking at the images (a, b), $(g_1(a), g_1(b))$, $(g_2(g_1(a)), g_2(g_1(b)))$, …and so on, by pigeonhole principle we can find a subsequence with $m \leq n^2$.

By appending n such sequences together, we can collapse all n inputs to the same output.

Remark: This problem comes up in the theory of deterministic finite automata, where good sequences are called "synchronizing words". See `https://en.wikipedia.org/wiki/Synchronizing_word`. □

7. Let ABC be an acute triangle with circumcircle γ and incenter I. Let D be the midpoint of minor arc $\widehat{BC}$ of γ. Let P be the reflection of the incenter of ABC over side BC. Suppose line DP meets γ again at a point Q on minor arc $\widehat{AB}$. Show that $AI = IQ$.

Solution. Let $R = \overline{DQ} \cap \overline{BC}$ and let T denote the foot of tangency from the incircle to $\overline{BC}$. Let $K = \overline{AD} \cap \overline{BC}$.



It's well-known that D is the circumcenter of $\triangle BIC$ (trillium theorem). Moreover, by the so-called shooting lemma, we have

$$DI^2 = DB^2 = DC^2 = DK \cdot DA = DQ \cdot DR.$$

Now, by the problem condition, ray $\overline{RTK}$ bisects $\angle PRI$. So let $\angle PRI = 2\theta$ and $\angle PRT = \theta$. Using the similar triangles, we can transfer this to

$$180^\circ - \angle QIA = \angle QID = \angle DRI = 2\theta$$

and

$$\angle QAI = \angle QAD = \angle DRK = \theta.$$

This is enough to imply $\triangle IAQ$ is isosceles, as desired. $\square$