Berkeley Math Circle: Monthly Contest 5 Solutions

1. An artist paints identical dragons on two circular discs of the same size. On the first disc, the dragon covers the center, but on the second it is not. Show that it is possible to cut the second disc into two pieces that can be reassembled so the dragon covers the center.

   Solution. Lay the second disc on top of the first disc so that the dragons are lined up, and cut off the piece of the second disc that hangs over the edge of the first disc. This piece is congruent to the piece of the first disc that is not covered by the second disc, so the two pieces can be glued together to form a disc with the dragon in the same place as on the first disc. □

2. Let n be a positive integer. Show that $2n+1$ and $4n^2+1$ are *relatively prime*, that is, their only common factor is 1.

   Solution. Any common factor of the two numbers would also have to divide

   $$(4n^2+1)-(2n+1)(2n-1)=(4n^2+1)-(4n^2-1)=2.$$

   But both numbers are odd, since they are 1 more than an even number, so they are not divisible by 2. Thus, their greatest common factor is 1. □

3. Let P be a polynomial with positive real coefficients. Prove that if

   $$P\left(\frac{1}{x}\right)\geq\frac{1}{P(x)}$$

   holds for $x=1$, then it holds for every $x>0$.

   Solution. Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, so $a_k>0$ for every k. Since the statement holds for $x=1$, $P(1)\geq 1$. Then by Cauchy-Schwarz,

   $$P(x)P\left(\frac{1}{x}\right)=\left(\sum_{k=0}^{n}(\sqrt{a_kx^k})^2\right)\left(\sum_{k=0}^{n}\left(\sqrt{\frac{a_k}{x^k}}\right)^2\right)\geq\left(\sum_{k=0}^{n}a_k\right)^2=P(1)^2\geq 1.$$

   □

4. Let $\overline{ST}$ be a chord of a circle ω which is not a diameter, and let A be a fixed point on $\overline{ST}$. For which point X on minor arc $\widehat{ST}$ is the length AX minimized?

   Solution. Extend the circular segment to make a whole circle, and let O be its center. Draw OA and let it meet the circle at X. Then the circle with center A and radius AX is tangent to the larger circle at A, and thus lies entirely inside it. Therefore, the distance from A to any other point on the circular arc is greater than AX, so X is the desired point. □

5. Show that $x^2 + y^2 = z^5 + z$ has infinitely many relatively prime integer solutions.

Solution. We use that there are infinitely many primes $p = 4k+1$, and that all these primes can be written as a sum of two squares. Now, let $z = p$ be such a prime. Then $p = a^2 + b^2$ for some integers a and b, so

$$z^5 + z = p(p^4 + 1) = (a^2 + b^2)(p^4 + 1) = (ap^2 + b)^2 + (bp^2 - a)^2.$$

Thus, we let $x = ap^2 + b$ and $y = bp^2 - a$, and x, y, and z are relatively prime since p is prime. □

6. Let $ABCD$ be a cyclic convex quadrilateral such that $AD + BC = AB$. Prove that the bisectors of the angles ADC and BCD meet on the line AB.

Solution. Let P be the point on side AB with $AP = AD$ and $BP = BC$, and suppose the circumcircle of $\triangle CPD$ intersects AB again at Q. WLOG assume P is closer to A than Q. Then $\angle CDQ = \angle QPC$, since P and D are inscribed in the same arc of circle CPD. But also, $\angle BPC = \angle BCP$, since triangle BCP is isosceles, and thus both these angles are $\frac{1}{2}(180^\circ - \angle ABC)$. But $\angle ADC = 180^\circ - \angle ABC$, since quadrilateral $ABCD$ is cyclic. Thus, $\angle QDC = \frac{1}{2}\angle ADC$, so Q lines on the angle bisector of $\angle ADC$. Similarly, Q also lies on the angle bisector of $\angle DCB$, so Q is the desired point. □

7. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc},$$

and that equality occurs if and only if $a = b = c = 1$.

Solution. Adding $3/(1+abc)$ to both sides, the desired inequality is equivalent to

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} + \frac{3}{1+abc} \geq \frac{6}{1+abc}.$$

We note that

$$\frac{1}{a(1+b)} + \frac{1}{1+abc} = \frac{1}{1+abc}\left(\frac{1+a}{a(1+b)} + \frac{b(1+c)}{1+b}\right),$$

$$\frac{1}{b(1+c)} + \frac{1}{1+abc} = \frac{1}{1+abc}\left(\frac{1+b}{b(1+c)} + \frac{c(1+a)}{1+c}\right),$$

$$\frac{1}{c(1+a)} + \frac{1}{1+abc} = \frac{1}{1+abc}\left(\frac{1+c}{c(1+a)} + \frac{a(1+b)}{1+a}\right).$$

Adding these three equations, we see that the six terms in the parentheses on the right pair up in three pairs of form $x + 1/x$ for some positive number x, which by AM-GM is at least 2. Thus, the terms in the parentheses sum to at least 6, and the desired inequality follows. □