Berkeley Math Circle: Monthly Contest 4 Solutions

1. A certain lecture has finitely many students and at least two students. Every student fell asleep exactly once and woke up exactly once. Suppose that for any two students, there was some time at which both were asleep. Prove that there was a time at which all the students were asleep.

   Solution. Let Alice be the first student to wake up. Then, immediately before Alice wakes up, all students must have been asleep, since every other student must have slept at the same time as Alice, and none of them have woken up yet. Thus, all the students are asleep. □

2. On a certain block, there are five houses in a line, which are to be painted red or green. If no two houses next to each other can be red, how many ways can you paint the houses?

   Solution. We break this into cases based on how many houses are red:

   Case 1: No houses are red. There is only one way to do this, since all the houses must be green.

   Case 2: One red house. There are 5 ways to choose the red house, and all the rest must be green.

   Case 3: Two red houses. We consider where the first red house is. If it is the first house on the block, then there are 3 choices for where the second red house is (the 3rd, 4th, or 5th house), if it is the 2nd house then there are 2 choices (the 4th or 5th house), and if it is the 3rd house there is only one choice (the 5th house). Thus, there are $3 + 2 + 1 = 6$ options in this case.

   Case 4: Three red houses. The only option is the 1st, 3rd, and 5th houses red and the other green.

   Adding these together, we get $1 + 5 + 6 + 1 = 13$ ways total. □

3. Let ABC be a triangle with incenter I. Show that the circumcenter of $\triangle BIC$ lies on the circumcircle of $\triangle ABC$.

   Solution. Let ray AI meet the circumcircle of $\triangle ABC$ again at a point L. We claim that $LB = LI = LC$, which will imply L is the circumcenter of $\triangle BIC$.

   Let us show $LB = LI$ (the proof $LC = LI$ is similar). It's enough to show $\triangle LIB$ is isosceles. Then

$$\begin{aligned}\angle IBL &= \angle IBC + \angle CBL = \frac{1}{2}\angle B + \angle CAL \\ &= \frac{1}{2}\angle B + \frac{1}{2}\angle A = \angle IAB + \angle ABI \\ &= 180^\circ - \angle AIB = \angle BIL\end{aligned}$$

as desired. $\square$

4. Let x, y, z be three real numbers. Prove the inequality

$$|x| + |y| + |z| - |x + y| - |y + z| - |z + x| + |x + y + z| \geq 0.$$

Solution. Let z be the greatest of the three numbers in absolute value. If $z = 0$, the left side is zero. Otherwise, dividing through by $|z|$, the left side becomes

$$\left|\frac{x}{z}\right| + \left|\frac{y}{z}\right| + 1 - \left|\frac{x+y}{z}\right| - \left(\frac{y}{z} + 1\right) - \left(1 + \frac{x}{z}\right) + \left|\frac{x+y}{z} + 1\right|.$$

Rearranging gives

$$\left(\left|\frac{x}{z}\right| + \left|\frac{y}{z}\right| - \left|\frac{x+y}{z}\right|\right) + \left(\left|\frac{x+y}{z} + 1\right| - \left(\frac{x+y}{z} + 1\right)\right).$$

The first term is nonnegative by the triangle inequality, and the second is nonnegative because it is the sum of a number and its absolute value. Thus, their sum is also nonnegative. $\square$

5. A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white and white becomes black). Determine the minimum number of mouse-clicks needed to make the chessboard all one color.

Solution. The minimum number of clicks needed is 98. One way to do this is to click all the even numbered rows one by one (49 clicks), then all the even numbered columns one by one (another 49 clicks). To show that this is minimal, note that there are $4 \cdot 97$ pairs of adjacent squares along the border, and that any such pair is mismatched. Each click can fix at most 4 adjacent pairs along the border, so it can reduce the number of mismatched pairs by at most 4, and thus at least 97 clicks are necessary. However, since the corners are not all the same color, some of the clicks also need to deal with corner squares and thus can only reduce the number of mismatched border pairs by 2. Thus, we need at least 98 clicks total, so this is indeed the minimum. $\square$

6. Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be nonnegative real numbers such that $x_i + y_i = 1$ for each $i = 1, 2, \ldots, n$. Prove that

$$(1 - x_1 x_2 \ldots x_n)^m + (1 - y_1^m)(1 - y_2^m) \ldots (1 - y_n^m) \geq 1,$$

where m is an arbitrary positive integer.

Solution. Suppose we have n coins that we flip m times each, where x_i is the probability the ith coin comes up heads and $y_i = 1 - x_i$ is the probability the ith coin comes up tails. Then the first term is the probability that every time we toss the n coins, at least one comes up tails, and the second term is the probability that each coin comes up heads at least once. But at least one of these events must occur, since if some coin never comes up heads, then it is never possible that all the coins came up heads, since that one didn't. Thus, the sum of the probabilities is at least 1. □

7. Let $P(x)$ be a polynomial with real coefficients so that $P(x) \geq 0$ for all real x. Prove that there exist polynomials $Q_1(x)$ and $Q_2(x)$ with real coefficients such that $P(x) = Q_1^2(x) + Q_2^2(x)$ for all x.

Solution. Since $P(x) \geq 0$ for all x, it can have no real roots except double roots, so we can write it as a product

$$P(x) = c \prod_{k=1}^{n} (x^2 + p_k x + q_k)$$

of quadratics with nonpositive discriminant, i.e. $p_k^2 - 4q_k \leq 0$. But then completing the square in each quadratic lets us write it as a sum of two squares of polynomials

$$x^2 + p_k x + q_k = \left(x_k + \frac{p_k}{2}\right)^2 + \left(\frac{\sqrt{p_k^2 - 4q_k}}{2}\right)^2.$$

Thus, $P(x)$ is a product of sums of two squares. But now we note that for any polynomials $A(x)$, $B(x)$, $C(x)$, $D(x)$ with real coefficients,

$$(A^2(x)+B^2(x))(C^2(x)+D^2(x)) = (A(x)C(x)+B(x)D(x))^2+(A(x)D(x)-B(x)C(x))^2,$$

i.e., a product of two sums of squares of polynomials is also a sum of two squares (Lagrange's identity for polynomials). Inductively applying this to the factors in $P(x)$ shows that $P(x)$ is also a sum of two squares. □