# Berkeley Math Circle: Monthly Contest 3 Solutions

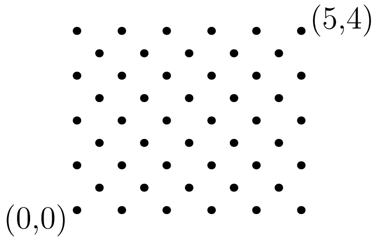
1. Determine whether there exist three positive integers $a$, $b$, $c$ such that $a+b$, $b+c$, and $c+a$ are all pairwise distinct prime numbers.

   *Solution.* No, there do not.

   First, if any of the primes is 2, say $a+b$, we must have $a=b=1$. But then $a+c=b+c$, violating the "pairwise distinct" hypothesis.

   Otherwise assume the prime numbers are $p$, $q$, $r$, each greater than 2. Then we get $p+q+r=2(a+b+c)$ but the left-hand side is even while the right-hand side is odd. □

2. Given integers $m \geq n \geq 1$, we define $F_{m,n}$ as the set of all points $(x,y)$ such that $0 \leq x \leq m$, $0 \leq y \leq n$, and $2x$, $2y$, and $x+y$ are all integers. For example, $F_{5,4}$ consists of 50 points and resembles the arrangement of stars on the American flag:




   (a) Find the number of points in $F_{m,n}$ in terms of $m$ and $n$.

   (b) Find all pairs $(m,n)$ such that $F_{m,n}$ has exactly 5000 points.

   *Solution.* Notice that the flag consists of two grids, an $(m+1)\times(n+1)$ grid and $m \times n$ grid. Therefore, $F_{m,n}$ consists of

$$(m+1)(n+1)+mn = 2mn+m+n+1 = \frac{1}{2}(2m+1)(2n+1)+\frac{1}{2}$$

   points.

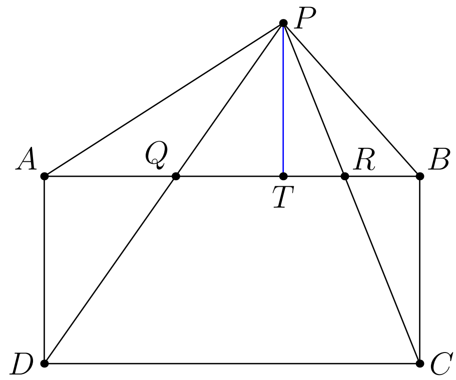
   If this equals 5000, then $(2m+1)(2n+1)=9999$. Factoring 9999 into all possible pairs of proper divisors and solving for $(m,n)$, we get five solutions:

$$(1666,1),(555,4),(454,5),(151,16),(50,49).$$

   □

3. Let $APBCD$ be a convex pentagon for which $ABCD$ is a square. Diagonals $PD$ and $AB$ meet at $Q$, while diagonals $PC$ and $AB$ meet at $R$. Prove that the sum of the areas of triangles $PAQ$ and $PBR$ equals the area of triangle $DQR$.

   *Solution.* Let $T$ be the foot from $P$ on line $AB$.




Since $\triangle AQD \sim \triangle TQP$, it follows that

$$AQ \cdot PT = QT \cdot AD.$$

The left-hand side is equal to twice the area of $\triangle PAQ$. The right-hand side is equal to twice the area of $\triangle QTD$. In other words

$$[PAQ] = [TQD].$$

Similarly,

$$[PRB] = [RTD].$$

Summing yields the result. □

4. If you label your thumbs with the number 1, index fingers with the number 2, and so on up to 5 on your little fingers, then when you put your hands together with each finger touching the corresponding finger on the you earn a score of

$$1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 4 + 5 \cdot 5 = 55$$

which is the highest score you can get. If you turn your hands so that one thumb is on the other index finger, and so on, you'd have $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + 5 \cdot 1 = 45$.

(a) By turning your hands in this way, what is the smallest score you can get?

(b) If aliens with 12 fingers on each hand play this game, what is their highest and lowest possible score across the 12 possible turns?

(c) If aliens with $n$ fingers on each hand play this game, what is their highest and lowest possible score across the $n$ possible turns?

*Solution.* The sum $1^2 + 2^2 + \cdots + n^2$ was maximal by rearrangement inequality, so we prove the lower bound. Suppose the finger touching the thumb is $k+1$, with $0 \le k < n$. Let $\ell = n - k$ for brevity, then the score is $A + B$ where

$$\begin{aligned} A &= 1(k+1) + 2(k+2) + \cdots + (n-k)n \\ &= k\,(1 + 2 + \cdots + \ell) + (1^2 + \cdots + (n-k)^2) \\ B &= (\ell+1) \cdot 1 + (\ell+2) \cdot 2 + \cdots + (\ell+k) \cdot k \\ &= -\ell((\ell+1) + \cdots + n) + \left((\ell+1)^2 + \cdots + n^2\right) \\ A + B &= k(1 + 2 + \cdots + \ell) - (n-k)((\ell+1) + \cdots + n) + \left(1^2 + \cdots + n^2\right) \\ &= k(1 + \cdots + n) - n\,((\ell+1) + \cdots + n) + \left(1^2 + \cdots + n^2\right) \\ &= k(1 + \cdots + n) + n(1 + \cdots + \ell) - n(1 + \cdots + n) + \left(1^2 + \cdots + n^2\right) \end{aligned}$$

The last two terms are constant, so we focus on

$$\begin{aligned} k(1+\cdots+n)+n(1+\cdots+\ell) &= \frac{n(n+1)}{2}\cdot(n-\ell)+n\cdot\frac{\ell(\ell+1)}{2} \\ &= \frac{n}{2}\left[(n+1)(n-\ell)+\ell(\ell+1)\right] \\ &= \frac{n}{2}\left[\ell^2-n\ell+(n^2+n)\right]. \end{aligned}$$

This is a quadratic, so the minimum is achieved when $\ell=\lfloor n/2\rfloor$. Substituting this in to the above expression gives the long expression above. □

5. Suppose $f$ is a function such that $f(xy+1)=xf(y)-f(x)+6$ for all real numbers $x$ and $y$. Find all possible functions $f$ that satisfy this equation and prove that no other functional solutions exist.

   *Solution.* Set $y=0$. Then $f(1)=xf(0)-f(x)+6$, which implies $f(x)=xf(0)-f(1)-6$, which in turn implies that $f$ is a linear function. Set $f(z)=az+b$. Then the functional equation implies $a(xy+1)+b=x(ay+b)-(ax+b)-6$. Since this must be true for all $x$ and $y$, one can equate similar terms: $a+b=-b+6$ and $bx-ax=0$. These imply $a=b$ and $a=b=2$. So $f(z)=2z+2$. Derivation shows that only one function $f$ exists. □

6. Let $ABC$ be a nondegenerate triangle. Let $A_1$, $B_1$, $C_1$ be any points on lines $BC$, $CA$, $AB$, respectively. Let $A_2$, $B_2$, $C_2$ denote the midpoinst of $AA_1$, $BB_1$, $CC_1$, respectively.

   Prove that the points $A_2$, $B_2$ and $C_2$ are collinear if and only if one or more of $A_1$, $B_1$ and $C_1$ coincides with a vertex of the triangle $ABC$.

   *Solution.* We proceed by barycentric coordinates on $\triangle ABC$. We let the concurrence point have coordinates $(u:v:w)$, so that $A_1=(0:v:w)$, $B_1=(u:0:w)$ and $C_1=(u:v:0)$. Since the $A$-midline has equation $y+z=x$ and similarly for the others, it follows that

$$\begin{aligned} A_2 &= (v+w:v:w) \\ B_2 &= (u:u+w:w) \\ C_2 &= (u:v:u+v). \end{aligned}$$

   The determinant equals

$$0=\det\begin{bmatrix} v+w & v & w \\ u & u+w & w \\ u & v & u+v \end{bmatrix}=3uvw$$

   and hence is zero iff at least one of $u$, $v$, $w$ is zero, as needed. □

7. Show that there are infinitely many pairs of integers $(x,y)$ satisfying

$$x^2+y^2+2017=2019xy.$$

*Solution.* Note that if $(x, y)$ with $1 \le x \le y$ then

$$(x, y) \to (y, 2019y - x)$$

and moreover the latter solution has $y \ge x$ and $2018y - x > y$. Thus starting from the solution $(1, 1)$ we may generate an infinite family of solutions. $\square$