Berkeley Math Circle: Monthly Contest 3 Solutions

1. How many ways are there to place three rooks on an 8×8 chessboard such that the rooks are in different columns and different rows?

   Solution. There are $\binom{8}{3}$ ways to pick which three of the eight columns we wish to pick the rooks in. Once the set of three columns is fixed, then we have eight choices for where to pick the rook in the leftmost column, then seven choices for where to pick the rook in the middle column, and finally six choices for where to pick the rook in the rightmost column. Thus, the answer is

   $$\binom{8}{3} \cdot 8 \cdot 7 \cdot 6 = 18816.$$

   □

2. Prove that if a, b, c, d are positive integers then $ab + bc + cd + da$ is not prime.

   Solution. Note that $ab + bc + cd + da = (a + c)(b + d)$ is the product of two integers each greater than 1. □

3. Alice and Bob play a game. There are 9 cards numbered 1 through 9 on a table, and the players alternate taking the cards, with Alice going first. A player wins if at any point they hold three cards with sum 15; if all nine cards are taken before this occurs, the game is a tie. Does either player have a winning strategy?

   Solution. Neither player has a winning strategy. Arrange the cards in the following 3×3 array:

   $$\begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix}$$

   In this setup, a sum of 15 corresponds to three-in-a-row. Thus we see that Alice and Bob are actually playing tic-tac-toe! (It is famously known that tic-tac-toe has no winning strategy; see for example `https://xkcd.com/832/`.) □

4. Let ABC be a triangle and P a point inside it. Lines AP, BP, CP meet the opposite sides at D, E, F. Assume that the three quadrilaterals $PDCE$, $PEAF$, $PFBD$ are all bicentric. Prove that triangle ABC is equilateral.

   (A quadrilateral is bicentric if it can be inscribed inside a circle, and it also can have a circle inscribed inside it.)

   Solution. First, from just the fact the three quadrilaterals are cyclic, we contend that P is the orthocenter of $\triangle ABC$. By power of a point, we have $AF \cdot AB = AP \cdot AD = AE \cdot AC$, hence quadrilateral $BFEC$ is cyclic as well.

   $$\measuredangle AEP = \measuredangle AFP = \measuredangle BFC = \measuredangle BEC = \measuredangle PEA$$

which implies $\overline{BPE} \perp \overline{AC}$; similarly $\overline{CPF} \perp \overline{AB}$ and so on.

Next we show that if the quadrilaterals also have an incircle, then P is the incenter of $\triangle ABC$. Now consider quadrilateral $PDCE$, which we now know has diameter $\overline{PC}$, hence $PD^2 + CD^2 = PE^2 + CE^2$. Since it has an incircle, we have $PD + CE = PE + CD$. Together, this is enough to imply $PD = PE$ and $CD = CE$ (i.e. that $PDCE$ is a kite). In a similar fashion we get $PD = PE = PF$. So P also is the incenter of $\triangle ABC$.

In conclusion P is both the orthocenter and incenter of $\triangle ABC$ which can only occur if $\triangle ABC$ is equilateral with P its center. □

5. For which positive integers n does the polynomial $P(X) = X^n + X^{n-1} + \cdots + 1$ have a real root?

   Solution. The answer is odd n only. For such odd n, one can take $X = -1$ as a real root.

   Suppose n is even. We claim P has no real roots. Indeed, note first $P(1) = n + 1$, so 1 is not a real root. But for any $x \neq 1$, we have

$$P(x) = x^n + \cdots + 1 = \frac{x^{n+1} - 1}{x - 1} \neq 0$$

   since $n + 1$ is odd implies $x^{n+1} \neq 1$. □

6. Let a and b be positive integers, and let A and B be finite disjoint sets of positive integers. Assume that for every $i \in A \cup B$, we have $i + a \in A$ or $i - b \in B$. Prove that $a|A| = b|B|$.

   Solution. We consider a directed graph G on $V = A \sqcup B$ such that for every $i \in A \sqcup B$,

   - if $i + a \in A$ we draw $i \to (i + a)$,
   - if $i - b \in B$ we draw $i \to (i - b)$.

   Thus every vertex has outdegree at least one. But since $A \cap B = \varnothing$, each vertex has indegree at most one. Hence the directed graph G is 1-regular (all vertices have indegree and outdegree equal to 1).

   In that case, G consists of disjoint cycles.

   If we follow a cycle, we gain $+a$ for each element of A and gain $-b$ for each element of B; it follows that in each cycle there are $\frac{b}{a}$ as many A-elements as B-elements. Summing over all cycles implies $a|A| = b|B|$. □

7. Find all ordered triples of non-negative integers (a, b, c) such that $a^2 + 2b + c$, $b^2 + 2c + a$, and $c^2 + 2a + b$ are all perfect squares.

   Solution. We have the trivial solutions $(a, b, c) = (0, 0, 0)$ and $(a, b, c) = (1, 1, 1)$, as well as the solution $(a, b, c) = (127, 106, 43)$ and its cyclic permutations.

The case $a = b = c = 0$ works. Without loss of generality, $a = \max\{a, b, c\}$. If b and c are both zero, it's obvious that we have no solution. So, via the inequality

$$a^2 < a^2 + 2b + c < (a + 2)^2$$

we find that $a^2 + 2b + c = (a + 1)^2 \implies 2a + 1 = 2b + c$. So,

$$a = b + \frac{c - 1}{2}.$$

Let $c = 2k + 1$ with $k \geq 0$; plugging into the given, we find that

$$b^2 + b + 2 + 5k \quad \text{and} \quad 4k^2 + 6k + 3b + 1$$

are both perfect squares. Multiplying both these quantities by 4, and setting $x = 2b + 1$ and $y = 4k + 3$, we find that

$$x^2 + 5y - 8 \quad \text{and} \quad y^2 + 6x - 11$$

are both even squares.

We may assume $x, y \geq 3$. We now have two cases, both of which aren't too bad:

- If $x \geq y$, then $x^2 < x^2 + 5y - 8 < (x + 3)^2$. Since the square is even, $x^2 + 5y - 8 = (x + 1)^2$. Then, $x = \frac{5y-9}{2}$ and we find that $y^2 + 15y - 38$ is an even square. Since $y^2 < y^2 + 15y - 38 < (y + 8)^2$, there are finitely many cases to check. The solutions are $(x, y) = (3, 3)$ and $(x, y) = (213, 87)$.
- Similarly, if $x \leq y$, then $y^2 < y^2 + 6x - 11 < (y + 3)^2$, so $y^2 + 6x - 11 = (y + 1)^2$. Then, $y = 3x - 6$ and we find that $x^2 + 15x - 38$ (!) is a perfect square. Amusingly, this is the exact same thing (whether this is just a coincidence due to me selecting the equality case to be $x = y$, I'm not sure). Here, the solutions are $(x, y) = (3, 3)$ and $(x, y) = (87, 255)$.

Converting back, we see the solutions are $(0, 0, 0)$, $(1, 1, 1)$ and $(127, 106, 43)$ and its cyclic permutations. $\square$