Berkeley Math Circle: Monthly Contest 2 Solutions

1. Do there exist five points in the plane, not all collinear, such that the distance between any pair is one of $\{1, 2, 3, \ldots 9\}$?

   Solution. Yes. Take the five points $(0, 0)$, $(\pm 3, 0)$ and $(0, \pm 4)$. The distances which appear are 3, 4, 5, 6, and 8. $\square$

2. Consider three hockey pucks lying on a level sheet of ice; the pucks are not collinear. In a move, one may select any of the pucks and hit it so that it passes through the midpoint of the other two pucks. Determine whether it is possible, after 2019 such moves, for all pucks to be in their original positions.

   Solution. It is not possible. Label the pucks A, B, C. Then the triangle ABC alternates between being oriented clockwise and counterclockwise between each move. Thus, the pucks cannot return to their original positions after an odd number of moves. $\square$

3. We use the digits $1, 2, \ldots, 9$ once each to form two integers (e.g., 7419 and 82635). What two integers formed in this way have the greatest product? Prove your answer.

   Solution. The answer is 9642×87531.

   Place value is an important feature of this problem, but it's awkward to write about since the leading digits matter most, yet we do not know how many digits each number will have. As a workaround, let us prepend "0." to the two integers we are forming, making them into decimals. For example, 7419 and 82635 would become 0.7419 and 0.82635. Regardless of how many digits our two integers have, the effect is to divide their product by 10^9. Thus, whichever decimals formed in this way have the largest product will correspond to the integers that have the largest product in the original formulation of the problem. The benefit of this transformation is that the leading digits now have a definite place value of tenths, etc.

   It is clear that the digits of each decimal should be in descending order (otherwise we can increase that number by rearranging them). Less obviously, a larger digit should never be assigned a lower place value in one decimal than the place value assigned to a smaller digit in the other decimal. For example, we should not form the decimals $0.98\ldots$ and $0.7\ldots$, where the larger 8 is assigned a place value of hundredths while the smaller 7 is assigned a place value of tenths. Proof: Let our two decimals be x and y, let digit d have place value 10^{-a} in decimal x, and let digit e have place value 10^{-b} in decimal y, where $d > e$ and $a > b$. Then swapping these two digits increases the product of the two decimals by $(d-e)(10^{-b}y - 10^{-a}x) = (d-e)(10^{-a})(10^{a-b}y - x)$. Since $0.1 \leq x, y < 1$, we have $y > 10^{-1}x$, and therefore the increase is positive.

   As a result of the foregoing, we see that the tenths digits of our two decimals must be 9 and 8 (in some order), the hundredths digits must be 7 and 6 (in some order), the thousandths must be 5 and 4, the ten-thousandths 3 and 2, and the hundred-thousandths 1 (and 0). Now the sum of the two decimals is fixed, so their product is maximized by making the two numbers as close to each other as possible. This is achieved by 0.9642 and 0.87531, and we are done. □

4. Let $ABCD$ be an isosceles trapezoid, and let E be the foot of the altitude from A to line BC. Prove that line DE passes through the centroid of $\triangle ABC$.

Solution. Let M be the midpoint of BC. Also, let F be the foot from D to BC. Then $AEFD$ is a rectangle. Define G as the intersection of AM and DE. Then $\triangle AGD \sim \triangle EMG$, and we get

$$\frac{GM}{GA} = \frac{ME}{DA} = \frac{ME}{FE} = \frac{1}{2}$$

which implies G is the centroid of $\triangle ABC$, since the centroid divides the A-median in a 2:1 ratio. Thus G lies on DE. □

5. Let n, k, r be positive integers. Suppose we have a collection of sets $S_1, S_2, \ldots, S_r$, where each S_i is a subset of $\{1, 2, \ldots, n\}$ consisting of one or more consecutive integers. We say that such a collection is a *k-fold perfect cover* of $\{1, 2, \ldots, n\}$ if each element of $\{1, 2, \ldots, n\}$ occurs in exactly k of the sets S_i. (As an example, $S_1 = \{1\}$, $S_2 = \{1, 2, 3\}$, $S_3 = \{2\}$, $S_4 = \{3, 4, 5\}$, $S_5 = \{4, 5\}$ is a 2-fold perfect cover when $n = 5$.)

Given a collection of sets which is a k-fold perfect cover of $\{1, 2, \ldots, n\}$, show that we can partition the collection into k subcollections, each of which is a 1-fold perfect cover of $\{1, 2, \ldots, n\}$.

Solution. We argue by contradiction. Fix n. Consider the smallest collection of S_i's (smallest as in least number of sets) that is a k-fold perfect cover for some k and can't be partitioned as desired. Choose the S_i of the form $\{1, 2, \ldots, m\}$ where m is as small as possible. If $m = n$, then remove S_i from the collection; the rest of the collection is a $(k-1)$-fold perfect cover and can be partitioned as desired, but this gives a partition of our k-fold perfect cover, so we have a contradiction. Hence we may assume $m < n$.

We claim that there is some S_i of the form $\{m+1, m+2, \ldots, m+m'\}$. Suppose there isn't. Then every S_i containing $m+1$ contains m. But it is also true that every set containing 1 contains m. Since m doesn't occur more often than 1 in our collection, it follows that every set containing $m+1$ contains 1. But there is a set containing 1 that doesn't contain $m+1$ (namely $\{1, 2, \ldots, m\}$), so we have a contradiction. This proves our claim.

Hence our collection contains two sets of the form $\{1, 2, \ldots, m\}$ and $\{m+1, m+2, \ldots, m+m'\}$. By replacing these sets with their union, we obtain a smaller collection, which can be partitioned as desired (because it is smaller than the smallest collection that can't be). This enables us to construct a partition of our original collection, so we again have a contradiction.

(This problem appears somewhat intimidating, but there is an appealing logic to the solution which can be explained by the following story. Imagine that a company has k employee positions. Jobs are in demand, so any vacancy at the company is filled immediately, though employees do not necessarily stay long. We regard the sets S_i as indicating the tenure of each person hired. Thus in the example stated in the problem, 5 people cycle through 2 positions; person 1 works for day 1 only, person 2 works from day 1 to day 3, etc. The fact that $S_1, \ldots, S_r$ form a k-fold perfect cover of $\{1, 2, \ldots, n\}$ expresses precisely that all positions are continuously filled from day 1 to day n.

But what does this mean? It means that when any number of people leave the company, the same number of people are instantly brought on. If we imagine each employee to have their own office, then the vacated offices can always be given to the newcomers; no employee need ever change offices. Each office has exactly one occupant at a time, so the S_i's corresponding to the employees who have occupied a given office form a 1-fold perfect cover of $\{1, 2, \ldots, n\}$, and this solves the problem.) $\square$

6. Prove that if the line joining the circumcenter O and the incenter I is parallel to side BC of an acute triangle, then $\cos B + \cos C = 1$.

Solution. We prove the following more general result, called *Carnot's theorem*: in a triangle ABC we have

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}$$

where r and R are the inradius and circumradius of $\triangle ABC$.

The altitude from vertex C divides side AB into two segments (one of which may be negative), giving $b\cos A + a\cos B = c$. The other two altitudes give $b\cos C + c\cos A = b$ and $c\cos B + b\cos C = a$.

Adding all there equations to $a\cos A + b\cos B + c\cos C$ gives

$$(a+b+c)(\cos A + \cos B + \cos C) = (a+b+c) + a\cos A + b\cos A + c\cos C$$

so

$$\cos A + \cos B + \cos C = 1 + \frac{a\cos A + b\cos B + c\cos C}{a+b+c}.$$

The area of ABC is $r(a+b+c)/2$ and from the 3 triangles into which circumradii divide the triangle $R(a\cos A + b\cos B + c\cos C)$. Hence $r/R = \frac{a\cos A + b\cos B + c\cos C}{a+b+c}$. Substituting this in, we have proved Carnot's theorem.

In the present problem, we have $\cos A = \frac{r}{R}$ since the distance from O to BC equals r. This implies the problem. $\square$

7. To play the lottery game Sum Thing, you choose five distinct numbers from 1 to 50, then the lottery master chooses five distinct numbers from 1 to 50. If there exist a nonempty subset of your five numbers and a nonempty subset of the lottery master's five numbers such that both subsets have the same sum, then you win.

Can you choose five numbers that guarantee a win? Either demonstrate such a set, with a proof of validity, or prove that no such set exists.

Solution. Yes, there is such a set. One such set is $\{4, 8, 16, 32, 42\}$.

With this ticket, the subsums are all multiples of 4 from 4 to 60, and all integers congruent to 2 (mod 4) from 42 to 102. So we must show that any five numbers chosen by the lottery master will have a subsum equal to one of these.

We partition the numbers from 1 to 50 into three types: evens (`E`), large odds (`O`), and small odds (`o`). An odd is "large" if it exceeds 20, otherwise "small". We argue by contradiction, starting from the assumption that the lottery master has found five numbers that will defeat the player.

First, some easy observations. Clearly any `E`'s chosen by the lottery master must be 2 (mod 4), and must be no larger than 38. The lottery master can't choose three `E`'s, since either two of them would add up to a multiple of 4 from 4 to 60, or all three would add up to an integers that is 2 (mod 4) from 42 to 102. It is also easy to see that any two `O`'s chosen by the lottery master must be in different classes (mod 4); in particular, the lottery master can't choose three `O`'s. Finally, any two `o`'s chosen by the lottery master must be in the same class mod 4, and the lottery master can't choose four `o`'s.

The lottery master must choose at least one `E`. The only other possibility not yet ruled out is `OOooo`, but in this case, the smallest `o` combines with whichever `O` is not in the same class modulo 4 to make a multiple of 4 less than or equal to 60.

Now we have two overlapping cases: the lottery master can have `Eoo??` or `EEoOO`.

- Case `Eoo??`: The `E` and two `o`'s will add up to a multiple of 4, which must be at least 64. This implies that `E` is 30, 34, or 38. We must avoid sums of the form $\texttt{o} + \texttt{O}$ that are 2 (mod 4) and ?42, but we must also avoid sums of the form $\texttt{E} + \texttt{o} + \texttt{O}$ that are 2 (mod 4) and ?102.
  - If any `o` is 19, then these two considerations rule out all `O` except 49 (which requires that the `E` is 38).
  - If any `o` is 17, then these considerations rule out all `O` except 21.
  - If any `o` is 15, then these considerations rule out all `O` except 23.
  - It isn't possible for all `o` to be less than 15.

  Since we have at least two `o`'s, we can't have any `O`. We already ruled out four `o`'s, so we must have `EEooo`. But the smaller `E` and the smallest two `o`'s will add up to a multiple of 4 that's at most 60, so this is a dead end.
- Case `EEoOO`: The two `E`'s must add up to at least 64, so one of them is at least 34. Pick the `O` that's not congruent to the `o` modulo 4. Add these together, then add the smaller of the two `E`'s. The result is 2 (mod 4), and is in the range 42 to 102, so we are finished.

□