Berkeley Math Circle: Monthly Contest 1 Solutions

1. Which is larger, $A = 20^{19^{20}}$ or $B = 19^{20^{19}}$? (Here, a^{b^c} means $a^{(b^c)}$ and not $(a^b)^c$.)

   Solution. The answer is that A is larger. First,

$$\begin{aligned}\left(\frac{19}{20}\right)^{20} &= 0.95^{20} = 0.9025^{10} > 0.9^{10} \\ &= 0.81^5 = 0.8^5 = 0.32768 > 0.05 = \frac{1}{20}\end{aligned}$$

   which implies that $19^{20} > 20^{19}$. Thus, $A = 20^{19^{20}} > 20^{20^{19}} > 19^{20^{19}} = B$, as desired. □

2. We wish to distribute 12 indistinguishable stones among 4 distinguishable boxes B_1, B_2, B_3, B_4. (It is permitted some boxes are empty.)

   (a) Over all ways to distribute the stones, what fraction of them have the property that the number of stones in every box is even?

   (b) Over all ways to distribute the stones, what fraction of them have the property that the number of stones in every box is odd?

   Solution. We will invoke the following famous theorem, colloquially called *stars and bars*: the number of ways to distribute a indistinguishable objects among b boxes is $\binom{a+b-1}{b-1}$. One can see this by considering a string of a objects ("stars") and $b-1$ dividers ("bars"), which give a distribution into boxes.

   The total number of ways to distribute the stones overall is equal to $\binom{12+3}{3} = 455$, by stars-and-bars argument.

   For (a), we imagine distributing the stones in pairs; we are distributing 6 double-stones. So the number of ways is $\binom{6+3}{3} = 84$. Thus the fraction is $\frac{84}{455} = \frac{12}{65}$.

   For (b), we also distribute the stones in pairs, but we need to place one stone in each box first: thus, we are distributing 4 double-stones (with the eight remaining stones). The number of ways to do this is $\binom{4+3}{3} = 35$. Thus the fraction is $\frac{35}{455} = \frac{1}{13}$. □

3. Find the number of ordered pairs (a, b) of positive integers such that a and b both divide 20^{19}, but ab does not.

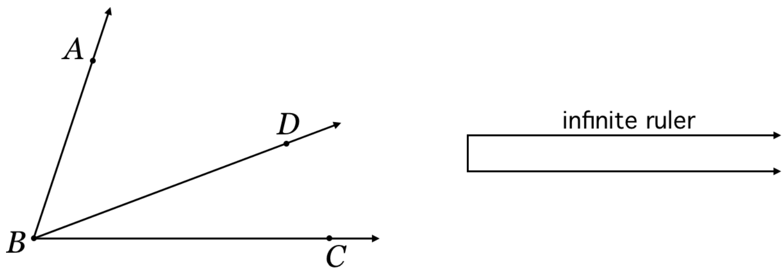
   Solution. Write $N = 20^{19} = 2^{38} \cdot 5^{19}$.

   The number of divisors of N is $39 \cdot 20 = 780$.

   The number of pairs (a, b) with ab dividing N is equal to the number of nonnegative integer solutions to $x + y \leq 38$ times the number of nonnegative integer solutions to $x + y \leq 19$; by a stars-and-bars argument this equals $\binom{40}{2}\binom{21}{2} = 163800$.

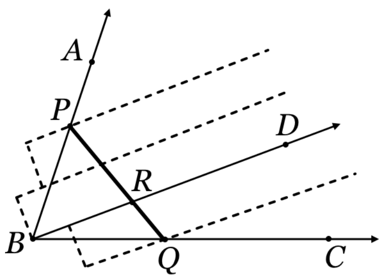
   Therefore the answer is $780^2 - 163800 = 444600$. □

4. An acute angle $\angle ABC$ and interior ray BD are given, as shown. Laura is given an *infinite ruler* which consists of two parallel rays joined on one end by a segment perpendicular to both of them. One may place the infinite ruler onto the diagram so that one of the infinite edges (marked with an arrow) passes through any two selected points in the diagram, or so that any edge of the ruler coincides with (i.e. exactly overlaps with) a portion of a segment, ray or line already in the diagram. Once the ruler is placed, one may draw any edge of the ruler "onto the diagram", in the usual fashion. One may also plot points where any two straight objects intersect.



Using only an infinite ruler, describe how to construct points P on ray BA and Q on ray BC such that ray BD intersects ray PQ at a point R with $PR = 2(QR)$. Then prove that your construction works.

Solution. Place the infinite ruler so that one infinite edge aligns with ray BD and the other infinite edge intersects ray BC, then trace its outline. Do the same thing twice on the other side of ray BD, so that the final edge intersects ray BA. These points of intersection give the desired segment PQ.



We outline the proof, which is mainly self-evident. The construction produces a set of four equally spaced rays, which therefore cut PQ into three congruent segments. It follows that $PR = 2(QR)$. □

5. Let a, b, c be positive real numbers. Assume that

$$\frac{a^{19}}{b^{19}} + \frac{b^{19}}{c^{19}} + \frac{c^{19}}{a^{19}} \leq \frac{a^{19}}{c^{19}} + \frac{b^{19}}{a^{19}} + \frac{c^{19}}{b^{19}}.$$

Prove that

$$\frac{a^{20}}{b^{20}} + \frac{b^{20}}{c^{20}} + \frac{c^{20}}{a^{20}} \leq \frac{a^{20}}{c^{20}} + \frac{b^{20}}{a^{20}} + \frac{c^{20}}{b^{20}}.$$

Solution. If we multiply the first equation by $(abc)^{19}$ then it can be rewritten as

$$(a^{19} - b^{19})(b^{19} - c^{19})(c^{19} - a^{19}) \leq 0.$$

Similarly, the desired equation is equivalent to

$$(a^{20} - b^{20})(b^{20} - c^{20})(c^{20} - a^{20}) \le 0.$$

These are evidently equivalent since $a^{19} - b^{19}$ and $a^{20} - b^{20}$ have the same sign, etc. $\square$

6. Let ABC be a triangle with circumcircle Γ, whose incircle touches BC, CA, AB at D, E, F. We draw a circle tangent to segment BC at D and to minor arc $\widehat{BC}$ of Γ at the point A_1. Define B_1 and C_1 in a similar way. Prove that lines A_1D, B_1E, C_1F are concurrent.

Solution. By the so-called "shooting lemma" (which is proved by taking homothety at A_1) we find that line A_1D passes through the arc midpoint of $\widehat{BAC}$ of Γ; denote this arc midpoint by X. Define Y and Z similarly, so that Y lies on line B_1E and Z lies on line C_1F.

We note the triangles XYZ and DEF are homothetic, since their corresponding sides are parallel: line YZ and EF are both known to be perpendicular to the internal $\angle A$-bisector. Thus DX, EY, FZ meet at a point — which thus is also the concurrency point of A_1D, B_1E, C_1F. $\square$

7. Are there positive integers a and b satisfying $a^2 - 23 = b^{11}$?

Solution. The answer is no. We may write the given equation as

$$a^2 + 45^2 = b^{11} + 2^{11}.$$

Taking modulo 4, we have $b^{11} \equiv a^2 + 45^2 \equiv a^2 + 1 \pmod 4$ which forces $b \equiv 1 \pmod 4$ and $a \equiv 0 \pmod 2$. Thus $b + 2 \equiv 3 \pmod 4$.

Let ν_p be the usual p-adic notation. Let p be an odd prime divisor of $b+2$ which is $3 \pmod 4$, and with $\nu_p(b+2)$ being odd (in particular nonzero). As a consequence of Fermat's Christmas theorem, we have

$$\nu_p(a^2 + 45^2) = \begin{cases} 4 & p = 3 \text{ and } \nu_3(a) \ge 2 \\ 2 & p = 3 \text{ and } \nu_3(a) = 1 \\ 2 & p = 5 \text{ and } \nu_5(a) \ge 1 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand for any $p \ne 11$ we have $\nu_p(b^{11}+2^{11}) = \nu_p(11)+\nu_p(b+2) = \nu_p(b+2)$, by the exponent lifting lemma (since we assumed $p \mid b+2$). This is odd, and hence we get a contradiction. $\square$