Berkeley Math Circle: Monthly Contest 8 Solutions

1. Consider the digits 0, 1, ..., 9. What is the largest subset S of these 10 digits we can find for which no two distinct digits in S have a prime sum?

   Solution. The answer is 5, obtained for example by $S = \{0, 2, 4, 6, 8\}$.

   To show that no larger set S is possible, note the pairings:

$$\begin{aligned} 3 &= 0 + 3 \\ 3 &= 1 + 2 \\ 11 &= 4 + 7 \\ 11 &= 5 + 6 \\ 17 &= 8 + 9. \end{aligned}$$

   If we had a set S of size at least six, we would pick both digits from one of the five pairs above. As 3, 11, 17 are all primes this shows that we cannot have $|S| \geq 6$. □

2. Prove that if the side lengths of a nondegenerate triangle are all powers of 2, then it is isosceles.

   Solution. Consider the longest side of the triangle, say $AB = 2^n$. We claim that one of AC and BC is equal to 2^n.

   Assume for contradiction this is false. Then both AC and BC are at most 2^{n-1}, but then the triangle inequality gives

$$2^n = AB < AC + BC \leq 2^{n-1} + 2^{n-1} = 2^n$$

   which is a contradiction. □

3. Let a, b, c be real numbers (not necessarily positive) for which $\min(a+b, b+c, c+a) \geq 2$. Show that

$$(a + b + c)^3 \geq a^3 + b^3 + c^3 + 24.$$

   Solution. This turns out to follow from the algebraic identity

$$(a + b + c)^3 - a^3 - b^3 - c^3 = 3(a + b)(b + c)(c + a) \geq 24.$$

   □

4. Let a, b, c be positive integers such that $a^2 - bc$ is a square. Prove that $2a + b + c$ is not prime.

Solution. Suppose that $a^2 - bc = d^2$, so that $(a-d)(a+d) = bc$. Then, by the so-called "factor lemma", we can find positive integers w, x, y, z such that $a - d = wx$, $a + d = yz$, $b = wy$, $c = xz$. Thus,

$$\begin{aligned} 2a + b + c &= (a-d) + (a+d) + b + c \\ &= wx + wy + xz + yz \\ &= (w+z)(x+y). \end{aligned}$$

which is clearly not prime. □

5. In an 100×100 chessboard two squares are *adjacent* if they share a common edge or vertex. Find the largest constant g with the following property: if we fill the chessboard with the numbers 1, 2, …, 10000 then we can find two adjacent squares whose labels differ by at least g.

   Solution. The answer is $g = 101$, with equality construction given by simply writing the numbers $1, 2, \dots, 10000$ in order (meaning the first row has the numbers $1, \dots, 100$ from left to right, the second row has the numbers $101, \dots, 200$ from left to right, and so on).

   We now prove that $g = 101$ is best possible. Given a labeling, consider the path from 1 to 10000. Note that this path takes at most 99 steps, and covers a "distance" of 9999. Thus some step must connect numbers differing by at least $\frac{9999}{99} = 101$. □

6. Prove that for any complex numbers z_1, z_2, …, z_n, satisfying $|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 = 1$, one can select $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{-1, 1\}$ such that

$$\left| \sum_{k=1}^{n} \varepsilon_k z_k \right| \le 1.$$

   Solution. Squaring and homogenizing, we will prove that one can pick the ε_i's such that

$$\left| \sum_{k=1}^{n} \varepsilon_k z_k \right|^2 \le S := \sum_{k=1}^{n} |z_k|^2 .$$

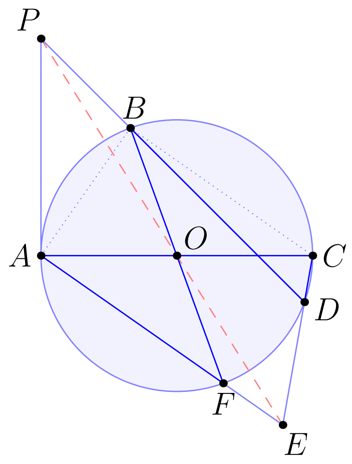
   To see this, the left-hand side equals

$$\left(\sum_{k=1}^{n} \varepsilon_k z_k \right) \left(\sum_{k=1}^{n} \varepsilon_k \overline{z_k} \right) = S + \sum_{i \ne j} \varepsilon_i \varepsilon_j z_i \overline{z_j}.$$

   Suppose we select the ε_i by coin flip. Then the expected value of each term $\varepsilon_i \varepsilon_j z_i \overline{z_j}$ is zero. Thus by *linearity of expectation* the expected value of the entire latter sum is zero. Thus there exists a choice of ε_i's for which the sum is non-positive, as desired. □

7. Let ω and O be the circumcircle and circumcenter of right triangle ABC with $\angle B = 90°$. Let P be any point on the tangent to ω at A other than A, and suppose ray PB intersects ω again at D. Point E lies on line CD such that $\overline{AE} \parallel \overline{BC}$. Prove that P, O, and E are collinear.

Solution. Let F be the point diametrically opposite B.



Apply Pascal theorem to $AAFBDC$ to finish. $\square$